Gauge Invariance of the Standard Model in the Causal Approach to Renormalization Theory

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Abstract

We present an extremely simple solution of the renormalization of gauge theories based on Epstein-Glaser approach to renormalization theory.

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1 Introduction

The causal approach to renormalization theory pioneered by Epstein and Glaser [21], [22] provides essential simplification at the fundamental level as well as at to the computational aspects. This is best illustrated in [38] where quantum electrodynamics is constructed entirely in the framework of the causal approach. Moreover, one can use the same ideas to analyse other theories as for instance, Yang-Mills theories [9] [10] [12] [13] [1] [2] [4]-[6], [29]-[32], [35] [37] [20], gravitation [23], [24], [42], etc.

Let us remind briefly the main ideas of Epstein-Glaser-Scharf approach. According to Bogoliubov and Shirkov, the S-matrix is constructed inductively order by order as a formal series of operator valued distributions:

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{\mathbb{R}^{4n}} dx_1 \cdots dx_n \, T_n(x_1, \cdots, x_n) g(x_1) \cdots g(x_n), \tag{1.0.1}$$

where $g(x)$ is a tempered test function in the Minkowski space $\mathbb{R}^4$ that switches the interaction and T_n are operator-valued distributions acting in the Fock space of some collection of free fields. These operator-valued distributions, which are called *chronological products* should verify some properties which can be argued starting from *Bogoliubov axioms*. These axioms will be detailed in the next Section. The main point is that one can show that, starting from a convenient *interaction Lagrangian* $T_1(x)$ one can construct the whole series $T_n, \quad n \geq 2$. The interaction Lagrangian must satisfy some requirements such like Poincaré invariance, hermiticity and causality; it is not easy to find a general solution of this problem but there are some rather general expressions fulfilling these demands, namely the so-called Wick polynomials. These are expressions operating in Hilbert spaces of a special kind, namely in Fock spaces. A Fock space is a canonical object attached to any single-particle Hilbert and reasonably describes a system of weakly interacting particles.

The physical S-matrix is obtained from $S(g)$ taking the *adiabatic limit* which is , loosely speaking the limit $g(x) \to 0$. One should also point out that the recursive process of constructing the chronological produces fixes them almost uniquely, more precisely the distribution T_n is unique up to a distribution $N_n(x_1, \cdots, x_n)$ with support in the set

$$\{(x_1, \cdots, x_n) \in (\mathbb{R}^4)^n | x_1 = \cdots = x_n\}.$$

This type of distribution are also called *finite renormalizations*.

In the old version of renormalization theory, one starts from the naive expressions of the chronological product and sees that they are not properly defined, i.e. some infinities do appear. The main obstacle is to amend the naive expression such that well defined expressions are obtained which do also verify Bogoliubov axioms. In Epstein-Glaser approach, the main problem of the the construction of the chronological products is done recurringly and it is reduced to the problem of distribution splitting. It can be proved that this operation has always solutions consistent with Bogoliubov axioms.

In the case of a gauge theory there is a supplementary property to be verified. The main obstacle in constructing the perturbation series for a gauge field is the fact that, as it happens for the electromagnetic field, one is forced to use non-physical degrees of freedom for the description of the free fields [45], [41], [34] in a Fock space formalism. One must consider an auxiliary Fock space $\mathcal{H}^{gh}$ including, beside the various fields, some fictious fields, called ghosts, and construct an supercharge that's it an operator Q verifying $Q^2 = 0$ such that the physical Hilbert space is $\mathcal{H}_{phys} \equiv Ker(Q)/Im(Q)$. The necessity to consider ghost fields comes mainly from the fact that, up to now, there is no other way to construct an interaction Lagrangian. On the other hand, one can construct a convenient interaction

Lagrangian in the bigger Hilbert space $\mathcal{H}_{gh}$ and apply the construction of Epstein and Glaser without any change. However, in this case one must impose, beside the usual Bogoliubov axioms, the supplementary condition that the S matrix factorizes to $\mathcal{H}_{phys}$. This condition proves to be too strong and one must replace it by a weaker condition of factorization to the physical Hilbert space in the adiabatic limit:

$$lim_{\epsilon\searrow 0}\int_{(\mathbb{R}^4)^{\times n}} dx_1\cdots dx_n g(\epsilon x_1)\cdots g(\epsilon x_n)[Q,T_n(x_1,\ldots,x_n)]_{Ker(Q)} = 0, \quad \forall n\geq 1. \tag{1.0.2}$$

Even this condition seems to be problematic because the adiabatic limit does not exists if zero-mass particles are present, so one must weaken further this requirement as it is done in [9] where one requires that:

$$[Q,T_n(x_1,\ldots,x_n)] = i\sum_{l=1}^{n}\frac{\partial}{\partial x_l^{\mu}}T^{\mu}_{n/l}(x_1,\ldots,x_n), \quad \forall n\in\mathbb{N}^* \tag{1.0.3}$$

for some Wick polynomials $T_{n/l}, \quad l=1,\ldots n$. This condition leads to the previous one if one formally takes the adiabatic limit and it is called the *gauge invariance* of the theory. It is impressive that this condition for $n=1$ and $n=2$ fixes almost uniquely the possible form of the interaction Lagrangian and leads to the presence of a r-dimensional Lie group of symmetries in the case of system composed of r Bosons of spin 1 [1], [25]. This result can be extended to the case of presence of matter fields [2], [26] paving the way to a rigorous understanding of the standard model of elementary particles. In [27] the analysis is pushed to order $n=3$ and the axial anomaly appears in a natural context, as an obstruction to the factorization condition (1.0.3) to the physical Hilbert space. The gauge invariance problem is now to prove that the identities (1.0.3) can be fulfilled for every $n\in\mathbb{N}$, more precisely to show that one can use the freedom left in the chronological products (the finite renormalizations) to impose gauge invariance in every order of perturbation theory. This problem is addressed in [38] in the case of quantum electrodynamics and in [10] [12] in the Yang-Mills case. The idea is to assume that one has (1.0.3) for $p=1,\ldots,n-1$ and prove it for $p=n$.

The main point of this paper is that with a proper formulation of the induction hypothesis one can simplify tremendously the proof such that it's mathematical rigor becomes obvious. The idea is that if one formulates the induction hypothesis in close analogy with the analysis of Epstein and Glaser then one can prove that in the order n one has instead of (1.0.3) the relation

$$[Q,T_n(x_1,\ldots,x_n)] = i\sum_{l=1}^{n}\frac{\partial}{\partial x_l^{\mu}}T^{\mu}_{n/l}(x_1,\ldots,x_n) + P_n(x_1,\ldots,x_n), \quad \forall n\in\mathbb{N}^* \tag{1.0.4}$$

where P_n is a finite renormalization. This finite renormalization can be so much restricted from the induction process that it is an elementary matter to show that one can modify appropriately the expressions T_n and $T_{n/l}$ in such a way that one has $P_n=0$. In this process, the use of some discrete symmetry like spatial inversion and charge conjugation is essential. In the case of the standard model these symmetries are no longer valid and one must replace them by the PCT covariance.

Moreover, our proof will show that there is no essential difference between the case of quantum electrodynamics and the more complicated case of the standard model: if one understands the first case then the more general one follows easily. We are able to give the generic form of a possible anomaly and show rigorously that it only can appear in order $n=3$, a fact well known in the literature.

We will adopt the gauge invariance in the form (1.0.3) so we will not touch the adiabatic limit problem in our analysis.

The paper is organized as follows. In the next Section we fix the notations and clarify the setting we use. We will present Bogoliubov axioms of the perturbation theory. Because the main point of our paper is to formulate the induction hypothesis in strict analogy to [21] we will summarize the induction argument used for a theory without gauge invariance. Then we present the modification of the setting one must impose to study quantum electrodynamics. In Section 3 we will give the proof of gauge invariance of quantum electrodynamics circumventing a certain problem from [38] connected with the use of some reduction formulæused in this reference. More details will be given at the proper place. In Section 4 we present the generalization of the whole procedure to the case of Yang-Mills fields with matter. Again, we simplify considerably the argument from [10]-[12]. The Conclusions are grouped in the last Section.

2 Perturbation Theory for QED

2.1 Bogoliubov Axioms

We give here the set of axioms imposed on the chronological products T_p following the notations of [21].

- First, it is clear that we can consider them *completely symmetrical* in all variables without loosing generality:

$$T_p(x_{\pi(1)}, \cdots x_{\pi(p)}) = T_p(x_1, \cdots x_p), \quad \forall \pi \in \mathcal{P}_p. \tag{2.1.1}$$

- Next, we must have *Poincaré invariance*. Because we will consider in an essential way Dirac fields, this amounts to suppose that in the in the Fock space we have an unitary representation $(a, A) \mapsto U_{a,A}$ of the group $inSL(2, \mathbb{C})$ (the universal covering group of the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$ - see [43] for notations) such that:

$$U_{a,A} T_p(x_1, \cdots, x_p) U_{a,A}^{-1} = T_p(\delta(A) \cdot x_1 + a, \cdots, \delta(A) \cdot x_p + a), \quad \forall A \in SL(2, \mathbb{C}), \forall a \in \mathbb{R}^4 \tag{2.1.2}$$

where $SL(2, \mathbb{C}) \ni A\delta(A) \in \mathcal{P}_+^\uparrow$ is the covering map. In particular, *translation invariance* is essential for implementing Epstein-Glaser scheme of renormalization.

Sometimes it is possible to supplement this axiom by corresponding invariance properties with respect to inversions (spatial and temporal) and charge conjugation. For the standard model only the PCT invariance is available.

- The central axiom seems to be the requirement of *causality* which can be written compactly as follows. Let us firstly introduce some standard notations. Denote by $V^+ \equiv \{x \in \mathbb{R}^4 | \quad x^2 > 0, \quad x_0 > 0\}$ and $V^- \equiv \{x \in \mathbb{R}^4 | \quad x^2 > 0, \quad x_0 < 0\}$ the upper (lower) lightcones and by $\overline{V^\pm}$ their closures. If $X \equiv \{x_1, \cdots, x_m\} \in \mathbb{R}^{4m}$ and $Y \equiv \{y_1, \cdots, y_n\} \in \mathbb{R}^{4m}$ are such that $x_i - y_j \notin \overline{V^-}, \quad \forall i = 1, \ldots, m, \quad j = 1, \ldots, n$ we use the notation $X \geq Y$. If $x_i - y_j \notin \overline{V^+} \cup \overline{V^-}, \quad \forall i = 1, \ldots, m, \quad j = 1, \ldots, n$ we use the notations: $X \sim Y$. We use the compact notation $T(X) \equiv T_p(x_1, \cdots, x_p)$ with the convention

$$T(\emptyset) \equiv \mathbf{1} \tag{2.1.3}$$

and by XY we mean the juxtaposition of the elements of X and Y. In particular, the expression $T(X_1X_2)$ makes sense because of the symmetry property (2.1.1). Then the causality axiom writes as follows:

$$T(X_1X_2) = T(X_1)T(X_2), \quad \forall X_1 \geq X_2. \tag{2.1.4}$$

Remark 2.1 *It is important to note that from (2.1.4) one can derive easily:*

$$[T(X_1), T(X_2)] = 0, \quad \text{if} \quad X_1 \sim X_2. \tag{2.1.5}$$

- The *unitarity* of the S-matrix can be most easily expressed (see [21]) if one introduces, the following formal series:

$$\bar{S}(g) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^{4n}} dx_1 \cdots dx_n \, \bar{T}_n(x_1, \cdots, x_n) g(x_1) \cdots g(x_n), \tag{2.1.6}$$

where, by definition:

$$(-1)^{|X|}\bar{T}(X) \equiv \sum_{r=1}^{|X|}(-1)^r \sum_{partitions} T(X_1)\cdots T(X_r); \tag{2.1.7}$$

here $X_1, \cdots, X_r$ is a partition of X, $|X|$ is the cardinal of the set X and the sum runs over all partitions. For instance, we have:

$$\bar{T}_1(x) = T_1(x) \tag{2.1.8}$$

and

$$\bar{T}_2(x,y) = -T_2(x,y) + T_1(x)T_1(y) + T_1(y)T_1(x). \tag{2.1.9}$$

One calls the operator-valued distributions $\bar{T}_n$ *anti-chronological products*. It is not very hard to prove that the series (2.1.6) is the inverse of the series (1.0.1) i.e. we have:

$$\bar{S}(g) = S(g)^{-1} \tag{2.1.10}$$

as formal series. Then the unitarity axiom is:

$$\bar{T}(X) = T(X)^\dagger, \quad \forall X. \tag{2.1.11}$$

Remark 2.2 *One can show that the following relations are identically verified:*

$$\sum(-1)^{|X|}T(X)\bar{T}(Y) = \sum(-1)^{|X|}\bar{T}(X)T(Y) = 0 \tag{2.1.12}$$

where the sum goes over all partitions $X \cup Y = \{1,\ldots,p\} \equiv Z, \quad X \cap Y = \emptyset$. *Also one has, similarly to (2.1.4):*

$$\bar{T}(X_1X_2) = \bar{T}(X_2)\bar{T}(X_1), \quad \forall X_1 \geq X_2. \tag{2.1.13}$$

A *renormalization theory* is the possibility to construct such a S-matrix starting from the first order term: $T_1(x)$ which is a Wick polynomial called *interaction Lagrangian* which should verify the following axioms:

$$U_{a,A}T_1(x)U_{a,A}^{-1} = T_1(\delta(A)\cdot x + a), \quad \forall A \in SL(2,\mathbb{C}), \tag{2.1.14}$$

$$[T_1(x), T_1(y)] = 0, \quad \forall x,y \in \mathbb{R}^4 \quad s.t. \quad x \sim y, \tag{2.1.15}$$

and

$$T_1(x)^\dagger = T_1(x). \tag{2.1.16}$$

Usually, these requirements are supplemented by covariance with respect to some discrete symmetries (like spatial and temporal inversions, or PCT), charge conjugations or global invariance with respect to some Lie group of symmetry.

It is not easy to find non-trivial solutions to the set of requirements (2.1.14), (2.1.15) and (2.1.16). In fact, this is a problem of constructive field theory. Fortunately, if one considers that the Hilbert space of the theory is of Fock type, then one has plenty of interesting solutions, namely the Wick polynomials. As underlined in the Introduction, this is one of the main reasons of extending the Hilbert space of a gauge system by including ghost fields: there is no other obvious solution of constructing the interaction Lagrangian without them.

Let us mention for the sake of the completeness the axioms connected with the *adiabatic limit*, although we will not use them, as we have said in the Introduction.

• Let us take in (1.0.1) $g \rightarrow g_\epsilon$ where $\epsilon \in \mathbb{R}_+$ and

$$g_\epsilon(x) \equiv g(\epsilon x). \tag{2.1.17}$$

Then one requires that the limit

$$S \equiv \lim_{\epsilon \searrow 0} S(g_\epsilon) \tag{2.1.18}$$

exists, in the weak sense, and is independent of the the test function g. In other words, the operator S should depend only on the *coupling constant* $g \equiv g(0)$. Equivalently, one requires that the limits

$$T_n \equiv \lim_{\epsilon \searrow 0} T_n(g_\epsilon^{\otimes n}), \quad n \geq 1 \tag{2.1.19}$$

exists, in the weak sense, and are independent of the test function g. One also calls the limit performed above, the *infrared limit*.

• Finally, one demands the *stability of the vacuum* and the *stability of the one-particle states* i.e.

$$\lim_{\epsilon \searrow 0} < \Phi, S(g_\epsilon)\Phi >= 1 \tag{2.1.20}$$

or,

$$\lim_{\epsilon \searrow 0} < \Phi, T_n(g_\epsilon^{\otimes n})\Phi >= 0, \quad \forall n \in \mathbb{N}^* \tag{2.1.21}$$

if Φ is the vacuum Φ_0 or any one-particle state.

These two requirement amount for the interaction Lagrangian to demand that

$$T_1 \equiv \lim_{\epsilon \searrow 0} T_1(g_\epsilon) \tag{2.1.22}$$

should exists, in the weak sense, and should be independent of the test function g. Moreover, we should have

$$< \Phi, T_1\Phi >= 0 \tag{2.1.23}$$

if Φ is the vacuum Φ_0 or any one-particle state.

2.2 Epstein-Glaser Induction

In this Subsection we summarize the steps of the inductive construction of Epstein and Glaser [21]. The main point is a careful formulation of the induction hypothesis. So, we suppose that we have the interaction Lagrangian $T_1(x)$ given by a Wick polynomial acting in a certain Fock space. The causality property (2.1.15) is then automatically fulfilled, but we must make sure that we also have (2.1.14) and (2.1.16).

We suppose that we have constructed the chronological products $T_p(x_1, \cdots, x_p), \quad p = 1, \ldots, n-1$ having the following properties: (2.1.1), (2.1.4) and (2.1.11) for $p \leq n-1$ and (2.1.4) and (2.1.5) for $|X_1| + |X_2| \leq n-1$. We want to construct the distribution-valued operators $T(X), \quad |X| = n$ such that the the properties above go from 1 to n.

Here are the main steps of the induction proof.

1. One constructs from $T(X), \quad |X| \leq n-1$ the expressions $\bar{T}(X), \quad |X| \leq n-1$ according to (2.1.7) and proves the properties (2.1.13) for $|X| + |Y| \leq n-1$ and (2.1.12) for $|Z| \leq n-1$.

2. **Lemma 2.3**

   Let us defines the expressions:

$$A'_n(x_1, \ldots, x_{n-1}; x_n) \equiv \sum{}' (-1)^{|Y|} T(X) \bar{T}(Y), \tag{2.2.1}$$

$$R'_n(x_1, \ldots, x_{n-1}; x_n) \equiv \sum{}' (-1)^{|Y|} \bar{T}(X) T(Y) \tag{2.2.2}$$

   where the sum $\sum'$ *goes over the partitions* $X \cup Y = \{1, \ldots, n\}, \quad X \cap Y = \emptyset, \quad Y \neq \emptyset, \quad x_n \in X.$ *Now, let us suppose that we have a partition* $P \cup Q = \{1, \ldots, n-1\}, \quad P \cap Q = \emptyset, \quad P \neq \emptyset.$

   Then:

   If $Qj \geq P$ *one has:*

$$A'_n(x_1, \ldots, x_{n-1}; x_n) = -T(Qn) T(P). \tag{2.2.3}$$

   and if $Qj \leq P$ *one has:*

$$R'_n(x_1, \ldots, x_{n-1}; x_n) = -T(P) T(Qn). \tag{2.2.4}$$

   The proof is elementary if one uses the causality properties (2.1.4) and (2.1.13).

3. **Corollary 2.4** *The expression*

$$D_n(x_1, \ldots, x_{n-1}; x_n) \equiv A'_n(x_1, \ldots, x_{n-1}; x_n) - R'_n(x_1, \ldots, x_{n-1}; x_n). \tag{2.2.5}$$

   have causal support i.e. supp$(D_n(x_1, \ldots, x_{n-1}; x_n)) \subset \Gamma^+(x_n) \cup \Gamma^-(x_n)$ *where we use standard notations:*

$$\Gamma^{\pm}(x_n) \equiv \{(x_1, \ldots, x_n) \in (\mathbb{R}^4)^n | x_i - x_n \in V^{\pm}, \quad \forall i = 1, \ldots, n-1\} \tag{2.2.6}$$

   The proof consists of noticing the local character of the support property and reducing all possible cases to typical situations from the preceding lemma.

4. We say that a numerical distribution $d(x_1, \ldots, x_{n-1}; x_n)$ is *factorizable* (or *disconnected*) if it can be written as: $d(X) = d_1(Y) d_2(Z)$ where Y, Z is a partition of X.

   Let us define the *degree* of a Wick monomial $deg(W)$ by assigning to every integer spin field factor and every derivative the value 1, for every half-integer spin field factor the value 3/2 and summing over all factors.

   Lemma 2.5 *The distribution $D_n(x_1, \ldots, x_{n-1}; x_n)$ can be written as a sum*

$$D_n(x_1, \ldots, x_{n-1}; x_n) = \sum_i d_i(x_1, \ldots, x_{n-1}; x_n) W_i(x_1, \ldots, x_{n-1}; x_n) \tag{2.2.7}$$

   where $W_i(x_1, \ldots, x_{n-1}; x_n)$ are linearly independent Wick monomials and $d_i(x_1, \ldots, x_{n-1}; x_n)$ are numerical distributions with causal support i.e $supp(d_i(x_1, \ldots, x_{n-1}; x_n)) \subset \Gamma^+(x_n) \cup \Gamma^-(x_n)$. Moreover, the set of Wick monomials appearing in the preceding formula can be obtained from the expression $T_1(x_1) \cdots T_1(x_n)$ by taking all possible Wick contractions, eliminating the monomials for for which the corresponding numerical distributions are factorizable and keeping a linearly independent set.

   Finally, the following limitations are valid:

$$\omega(d_i) + deg(W_i) \leq deg(T_1), \quad \forall i. \tag{2.2.8}$$

   The proof goes by induction.

5. If $d = (d_i)_{i=1}^N$ is a multi-component distribution and $SL(2, \mathbb{C}) \ni A \rightarrow D(A)$ is an N-dimensional representation of the group $SL(2, \mathbb{C})$ we define a new distribution according to:

$$(A \cdot d)(x) \equiv D(A) d(\delta(A^{-1}) \cdot x) \tag{2.2.9}$$

   and say that the distribution d is $SL(2, \mathbb{C})$-*covariant iff* it verifies:

$$A \cdot d = d. \tag{2.2.10}$$

   We remark that we have defined a $SL(2, \mathbb{C})$ action:

$$(A_1 \cdot A_2) \cdot d = A_1 \cdot (A_2 \cdot d), \quad \mathbf{1} \cdot d = d. \tag{2.2.11}$$

   For such multi-component distribution, the order of singularity $\omega(d)$ is, by the definition, the maximum of the orders of singularities of the components.

   Lemma 2.6 *The distributions $d_i(x_1, \ldots, x_{n-1}; x_n)$ defined above are $SL(2, \mathbb{C})$-covariant.*

   The proof follows from the induction hypothesis (2.1.2).

6. Now we have the following result from [10], [38]:

Lemma 2.7 *Let d be a $SL(2,\mathbb{C})$-covariant distribution with causal support. Then, there exists a causal splitting*

$$d = a - r, \quad supp(a) \subset \Gamma^+(x_n), \quad supp(r) \subset \Gamma^-(x_n) \tag{2.2.12}$$

which is also $SL(2,\mathbb{C})$-covariant and such that

$$\omega(a) \leq \omega(d), \quad \omega(r) \leq \omega(d). \tag{2.2.13}$$

We outline the proof because the argument is generic and it will also be used for the more general case of gauge invariance. It is known from the general theory of distribution splitting that there exists a causal splitting $d = a - r$ preserving the order of singularity. Then $A \cdot d = A \cdot a - A \cdot r$ is a causal splitting of the distribution $A \cdot d$. Because, by hypothesis, we have $A \cdot d = d$ it follows that we have

$$A \cdot a - a = A \cdot r - r. \tag{2.2.14}$$

But the left hand side has support in $\Gamma^+(x_n)$ and the right hand side in $\Gamma^-(x_n)$ so, the common value, denoted by P_A have the support in $\Gamma^+(x_n) \cap \Gamma^-(x_n) = \{(x_1, \cdots, x_n) \in (\mathbb{R}^4)^n | x_1 = \cdots = x_n\}$. But in this case, it is known from the general distribution theory that P_A is of the form

$$P_A(x) = p(\partial)\delta^{n-1}(x) \tag{2.2.15}$$

where

$$\delta^{n-1}(x) \equiv \delta(x_1 - x_n) \cdots \delta(x_{n-1} - x_n) \tag{2.2.16}$$

and p is a polynomial in the derivatives of maximal order $\omega(d)$. In particular, if $\omega(d) < 0$ we have $p = 0$ and the causal splitting is $SL(2,\mathbb{C})$-covariant. If $\omega(d) \geq 0$ then we easily derive that P_A verifies the following identity:

$$P_{A_1 \cdot A_2} = P_{A_1} + A_1 \cdot P_{A_2}. \tag{2.2.17}$$

This relation says that the map $A \rightarrow P_A$ is a $SL(2,\mathbb{C})$-cocycle with values in the finite dimensional space of polynomials of order not greater than $\omega(d)$. Because $SL(2,\mathbb{C})$ is a connected, simply connected and simple Lie group we can apply Hochschild lemma [43] and obtain that P_A is of the form

$$P_A = A_1 \cdot Q - Q \tag{2.2.18}$$

for some polynomial Q of order not greater than $\omega(d)$. In particular, we have

$$A \cdot (a - Q) = a - Q \tag{2.2.19}$$

so we have a $SL(2,\mathbb{C})$-covariant causal splitting $d = (a - Q) - (r - Q)$. ■

7. **Corollary 2.8** *There exists a $SL(2,\mathbb{C})$-covariant causal splitting:*

$$D_n(x_1, \ldots, x_{n-1}; x_n) = A_n(x_1, \ldots, x_{n-1}; x_n) - R_n(x_1, \ldots, x_{n-1}; x_n) \tag{2.2.20}$$

with supp($A_n(x_1, \ldots, x_{n-1}; x_n)$) $\subset \Gamma^+(x_n)$ and supp($R_n(x_1, \ldots, x_{n-1}; x_n)$) $\subset \Gamma^-(x_n)$.

For that reason, the expressions A_n and R_n are called *advanced* (resp. *retarded*) products.

8. **Lemma 2.9** *The following relation is true*

$$D_n(x_1,\ldots,x_{n-1};x_n)^\dagger = (-1)^{n-1} D_n(x_1,\ldots,x_{n-1};x_n). \tag{2.2.21}$$

In particular the causal splitting obtained above can be chosen such that

$$A_n(x_1,\ldots,x_{n-1};x_n)^\dagger = (-1)^{n-1} A_n(x_1,\ldots,x_{n-1};x_n). \tag{2.2.22}$$

The first assertion follows by elementary computations starting directly from the definition (2.2.5) and using the unitarity induction hypothesis (2.1.11) and the relations (2.1.12). This proves that by performing the substitutions:

$$\begin{aligned} A_n(x_1,\ldots,x_{n-1};x_n) &\to \frac{1}{2}\left[A_n(x_1,\ldots,x_{n-1};x_n)^\dagger + (-1)^{n-1}A_n(x_1,\ldots,x_{n-1};x_n)\right] \\ R_n(x_1,\ldots,x_{n-1};x_n) &\to \frac{1}{2}\left[R_n(x_1,\ldots,x_{n-1};x_n)^\dagger + (-1)^{n-1}R_n(x_1,\ldots,x_{n-1};x_n)\right] \end{aligned} \tag{2.2.23}$$

we do not affect the relation from the preceding corollary and we obtain a causal splitting verifying the relation from the statement without spoiling the $SL(2,\mathbb{C})$-covariance. ∎

9. Now we have

Theorem 2.10 *Let us define*

$$\begin{aligned} T_n(x_1,\cdots,x_n) &\equiv A_n(x_1,\cdots,x_{n-1};x_n) - A'_n(x_1,\cdots,x_{n-1};x_n) \\ &\equiv R_n(x_1,\cdots,x_{n-1};x_n) - R'_n(x_1,\cdots,x_{n-1};x_n). \end{aligned} \tag{2.2.24}$$

Then these expressions satisfy the $SL(2,\mathbb{C})$-covariance, causality and unitarity conditions (2.1.2) (2.1.4) (2.1.5) and (2.1.11) for $p = n$. If we substitute

$$T_n(x_1,\cdots,x_n) \to \frac{1}{n!}\sum_\pi T_n(x_{\pi(1)},\cdots,x_{\pi(n)}) \tag{2.2.25}$$

where the sum runs over all permutations of the numbers $\{1,\ldots,n\}$ then we also have the symmetry axiom (2.1.1). The generic expression of the chronological product is similar to that appearing in lemma 2.5

$$T_n(x_1,\ldots,x_n) = \sum_i t_i(x_1,\ldots,x_{n-1};x_n) W_i(x_1,\ldots,x_{n-1};x_n) \tag{2.2.26}$$

with the same limitation (2.2.8) on the numerical distributions:

$$\omega(t_i) + deg(W_i) \le deg(T_1), \quad \forall i. \tag{2.2.27}$$

The $SL(2,\mathbb{C})$-covariance is obvious. The causality axiom (2.1.4) follows from the two expressions of the definition of T_n if one takes into account the support properties of the advanced and retarded product and also uses lemma 2.3. The property (2.1.5) follows from general properties of the Wick monomials. The unitarity axiom is a result of the definition given above, the property of the advanced products from the preceding lemma, the expressions A'_n and the induction hypothesis (2.1.11) for $p \le n-1$. The symmetrization process is obvious. ∎

As we have mentioned in the Introduction the solution of the renormalization problem is not unique. The non-uniqueness is given by the possibility of adding to the distributions T_n some finite renormalizations N_n. There are some restrictions on these finite renormalizations coming from the Poincaré invariance and unitarity but still there remains some arbitrariness. One can restrict even further the arbitrariness requiring the existence of the adiabatic limit. One can prove that this limit does exists if there are no zero-mass particles in the spectrum of the energy-momentum quadri-vector.

2.3 Perturbation Theory for Zero-Mass Particles

We remind the basic facts about the quantization of the photon; for more details see [25] and references quoted there. Let us denote the Hilbert space of the photon by H_{photon}; it carries the unitary representation of the orthochronous Poincaré group $\mathsf{H}^{[0,1]} \oplus \mathsf{H}^{[0,-1]}$ (see [43]).

The Hilbert space of the multi-photon system should be, according to the basic principles of the second quantization, the associated symmetric Fock space $\mathcal{F}_{photon} \equiv \mathcal{F}^{+}(\mathrm{H}_{photon})$. One can construct in a rather convenient way this Fock space in the spirit of algebraic quantum field theory. One considers the Hilbert space $\mathcal{H}^{gh}$ generated by applying on the vacuum Φ_0 the free fields $A^{\mu}(x), \quad u(x), \quad \tilde{u}(x)$ called the *electromagnetic potential* (reps. *ghosts*) which are completely characterize by the following properties:

- Equation of motion:
$$\square A^{\mu}(x) = 0, \quad \square u(x) = 0, \quad \square \tilde{u}(x) = 0. \tag{2.3.1}$$

- Canonical (anti)commutation relations:
$$\begin{aligned} &[A^{\mu}(x), A^{\rho}(y)] = -g^{\mu\rho} D_0(x-y) \times \mathbf{1}, \quad [A^{\mu}(x), u(y)] = 0, \quad [A^{\mu}(x), \tilde{u}(y)] = 0 \\ &\{u(x), u(y)\} = 0, \quad \{\tilde{u}(x), \tilde{u}(y)\} = 0, \quad \{u(x), \tilde{u}(y)\} = D_0(x-y) \times \mathbf{1}; \end{aligned} \tag{2.3.2}$$

  here $D_m, \quad m \geq 0$ is the Pauli-Jordan distribution:
$$D_m(x) \equiv \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{2\sqrt{\mathbf{p}^2 + m^2}} exp(-ix_0\sqrt{\mathbf{p}^2 + m^2} + i\mathbf{x} \cdot \mathbf{p}). \tag{2.3.3}$$

- Covariance properties with respect to the Poincaré group. Let I_s and I_t be the space (time) inversion in the Minkowski space $\mathbb{R}^4$. Let $U_{a,A}, \quad U_{I_s}$ be the unitary operators realizing the $SL(2,\mathbb{C})$ transformations and the spatial inversion respectively and U_{I_t} the anti-unitary operator realizing the temporal inversion; then we require:
$$\begin{aligned} U_{a,A} A^{\mu}(x) U_{a,A}^{-1} &= \delta(A^{-1})^{\mu}{}_{\nu} A^{\nu}(\delta(A) \cdot x + a), \\ U_{a,A} u(x) U_{a,A}^{-1} = u(\delta(A) \cdot x + a), \quad U_{a,A} \tilde{u}(x) U_{a,A}^{-1} &= \tilde{u}(\delta(A) \cdot x + a) \end{aligned} \tag{2.3.4}$$

$$U_{I_s} A^{\mu}(x) U_{I_s}^{-1} = (I_t)^{\mu}{}_{\nu} A^{\nu}(I_s \cdot x), \quad U_{I_s} u(x) U_{I_s}^{-1} = u(I_s \cdot x), \quad U_{I_s} \tilde{u}(x) U_{I_s}^{-1} = \tilde{u}(I_s \cdot x); \tag{2.3.5}$$

$$U_{I_t} A^{\mu}(x) U_{I_t}^{-1} = (I_t)^{\mu}{}_{\nu} A^{\nu}(I_t \cdot x), \quad U_{I_t} u(x) U_{I_t}^{-1} = u(I_t \cdot x), \quad U_{I_t} \tilde{u}(x) U_{I_t}^{-1} = \tilde{u}(I_t \cdot x). \tag{2.3.6}$$

  The spatio-temporal inversion is: $U_{I_{st}} \equiv U_{I_s}\, U_{I_t}$.

- Charge invariance. The unitary operator realizing the charge conjugation verifies:

$$U_C A^\mu(x) U_C^{-1} = -A^\mu(x), \quad U_C u(x) U_C^{-1} = -u(x), \quad U_C \tilde{u}(x) U_C^{-1} = -\tilde{u}(x). \tag{2.3.7}$$

- Moreover, we suppose that these operators are leaving the vacuum invariant:

$$U_{a,A}\Phi_0 = \Phi_0, \quad U_{I_s}\Phi_0 = \Phi_0, \quad U_{I_t}\Phi_0 = \Phi_0, \quad U_C\Phi_0 = \Phi_0. \tag{2.3.8}$$

Remark 2.11 *One can easily prove that the operators $U_{a,A}$, U_{I_s} and U_{I_t} are realizing a projective representation of the Poincaré group i.e. they have suitable commutation properties (see [43] rel. (196) from ch. IX. 6). Also the charge conjugation operator commutes with these operators. (As it is well known, there is some freedom in choosing some phases in the definitions of the spatial and temporal inversions [43]; we have made the convenient choice which ensures this commutativity property).*

We suppose that in $\mathcal{H}^{gh}$ we have, beside the scalar product, a sesqui-linear form $< \cdot, \cdot >$ and we denote the conjugate of the operator O with respect to this form by $O^\dagger$. One can completely characterize this form by requiring:

$$A_\mu(x)^\dagger = A_\mu(x), \quad u(x)^\dagger = u(x), \quad \tilde{u}(x)^\dagger = -\tilde{u}(x). \tag{2.3.9}$$

Now, we define in $\mathcal{H}^{gh}$ an important operator called *supercharge* according to:

$$Q = \int_{\mathbb{R}^3} d^3x \partial^\mu A_\mu(x) \overleftrightarrow{\partial_0} u(x) \tag{2.3.10}$$

and one can prove the following properties:

$$Q\Phi_0 = 0 \tag{2.3.11}$$

and

$$\{Q, u(x)\} = 0, \quad \{Q, \tilde{u}(x)\} = -i\partial^\mu A_\mu(x), \quad [Q, A_\mu(x)] = i\partial_\mu u(x). \tag{2.3.12}$$

From these properties one can derive

$$Q^2 = 0; \tag{2.3.13}$$

so we also have

$$Im(Q) \subset Ker(Q). \tag{2.3.14}$$

Finally we have:

$$U_g Q = Q U_g, \quad \forall g = (a, A), I_s, I_t, C. \tag{2.3.15}$$

Then we have the central result

Theorem 2.12 *The sesqui-linear form $< \cdot, \cdot >$ factorizes to a well-defined scalar product on the completion of the factor space $Ker(Q)/Im(Q)$. Then there exists the following Hilbert spaces isomorphism:*

$$\overline{Ker(Q)/Im(Q)} \simeq \mathcal{F}_{photon}; \tag{2.3.16}$$

The representation of the Poincaré group and the charge conjugation operator are factorizing to $Ker(Q)/Im(Q)$ and are producing unitary operators with the exception of the temporal (and spatio-temporal) inversions which are anti-unitary.

Next, we denote by $\mathcal{W}$ the linear space of all Wick monomials acting in the Fock space $\mathcal{H}^{gh}$ generated by the fields $A_\mu(x),\ u(x)$ and $\tilde{u}(x)$.

Remark 2.13 *We notice that usually one constructs Wick monomials by first decomposing every the free fields in a creation and an annihilation parts and then ordering the creation parts to the left with respect to the annihilation parts (with the corresponding Jordan sign if Fermion fields are present). However, there is a way to define Wick monomials without this decomposition, by a substraction procedure [40] pg. 104; for instance:*

$$: A^\mu(x) A^\nu(x) :\equiv lim_{x_1,x_2\to x}[A^\mu(x_1)A^\nu(x_2) - (\Phi_0, A^\mu(x_1)A^\nu(x_2)\Phi_0)] \tag{2.3.17}$$

If M is such a Wick monomial, we define by $gh_\pm(M)$ the degree in u (resp. in $\tilde{u}$). The *ghost number* is, by definition, the expression:

$$gh(M) \equiv gh_+(M) - gh_-(M). \tag{2.3.18}$$

Then we define the operator:

$$d_Q M \equiv : QM : -(-1)^{gh(M)} : MQ : \tag{2.3.19}$$

on monomials M and extend it by linearity to the whole $\mathcal{W}$. The operator $d_Q : \mathcal{W} \to \mathcal{W}$ is called the *BRST operator*; its properties are following elementary from the properties of the supercharge: beside the Leibnitz rule we have:

$$d_Q u = 0, \quad d_Q \tilde{u} = -i\partial^\mu A_\mu, \quad d_Q A_\mu = i\partial_\mu u \tag{2.3.20}$$

and is a linearized version of the usual BRST transform [44]. Nevertheless it verifies:

$$d_Q^2 = 0. \tag{2.3.21}$$

We remind that if O is a self-adjoint operator verifying the condition

$$d_Q O = 0 \tag{2.3.22}$$

then it induces a well defined operator $[O]$ on the factor space $\overline{Ker(Q)/Im(Q)} \simeq \mathcal{F}_{photon}$. This kind of observables on the physical space are called *gauge invariant observables*. However, the operators of the type $d_Q O$ are inducing a null operator on the factor space, so are not interesting.

Usually one has to add into the game *matter* fields. These are operators for which one has to give separately the corresponding canonical (anti)commutation relations and transformation rules with respect to the Poincaré group and charge conjugation. By definition, we keep the same expression for the supercharge and construct the physical Hilbert space by the same factorization procedure. In particular, this will mean that the BRST operator acts trivially on the matter fields.

We can formulate now what we mean by a perturbation theory of electromagnetism + matter. By definition, this means that we can construct in $\mathcal{H}^{gh}$ the set of chronological products T_n as in the Subsection 2.1 and we impose in addition a factorization condition to the physical Hilbert space. To avoid infra-red divergence problems, we adopt as said in the Introduction the condition (1.0.3) which we prefer to write into the form:

$$d_Q T_n(x_1,\ldots,x_n) = i\sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} T^\mu_{n/l}(x_1,\ldots,x_n), \quad \forall n \in \mathbb{N}^* \tag{2.3.23}$$

for some Wick polynomials $T_{n/l}, \quad l = 1,\ldots n$.

By definition, this is the *gauge invariance* condition. It can be connected with the usual approaches based on the Ward identities imposed on the (renormalized) Feynman distributions.

Let us note that the Wick polynomials $T_{n/l}, \quad l = 1,\ldots n$, if they exists, are highly non-unique.

3 Renormalizability of Quantum Electrodynamics

3.1 The Interaction Lagrangian

By definition, in this case the matter field is a Dirac field of mass m denoted by $\psi(x) = \psi_\alpha(x)_{\alpha=1}^4$. To describe this field we need Dirac matrices $\gamma^\mu, \quad \mu = 0,\dots,3$ for which we prefer the chiral representation [43]:

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i = 1,2,3; \tag{3.1.1}$$

here $\sigma_i, \quad i = 1,2,3$ are the Pauli matrices. This is a representations in which the matrix $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$ is diagonal:

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1.2}$$

We denote as usual $\bar\psi(x) \equiv \psi(x)^*\gamma_0$; it is convenient to consider ψ ($\bar\psi$) as a column (line) vector. As before, the Dirac field is characterized by:

- Equation of motion (which is, of course the *Dirac equation*):

$$(i\gamma\cdot\partial + m)\psi(x) = 0. \tag{3.1.3}$$

- Canonical (anti)commutation relations:

$$\begin{gathered}[\psi(x), A^\mu(y)] = 0, \quad [\psi(x), u(y)] = 0, \quad [\psi(x), \tilde u(y)] = 0 \\ \{\psi(x), \psi(y)\} = 0, \quad \{\psi(x), \bar\psi(y)\} = S_m(x-y)\times\mathbf{1};\end{gathered} \tag{3.1.4}$$

  here $S_m, \quad m \geq 0$ is a 4×4 matrix given by:

$$S_m(x) \equiv (i\gamma\cdot\partial + m)D_m(x). \tag{3.1.5}$$

- Covariance properties with respect to the Poincaré group:

$$\begin{gathered}U_{a,A}\psi(x)U_{a,A}^{-1} = S(A^{-1})\psi(\delta(A)\cdot x + a), \\ U_{I_s}\psi(x)U_{I_s}^{-1} = i\gamma_0\psi(I_s\cdot x), \quad U_{I_t}\psi(x)U_{I_t}^{-1} = -i\gamma_1\gamma_3\psi(I_s\cdot x);\end{gathered} \tag{3.1.6}$$

  here

$$S(A) \equiv \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^* \end{pmatrix}. \tag{3.1.7}$$

- Charge invariance. The unitary operator realizing the charge conjugation verifies:

$$U_C\psi(x)U_C^{-1} = \gamma_0\gamma_2\bar\psi(x)^t. \tag{3.1.8}$$

These relations should be added to the ones from the preceding Subsection. It can be proved that Remark 2.11 stays true.

By definition, the interaction Lagrangian is:

$$T_1(x) \equiv e : \bar\psi(x)\gamma_\mu\psi(x) : A^\mu(x) \tag{3.1.9}$$

(here e is the electron charge) and one can verify easily that the properties (2.1.14), (2.1.15) and (2.1.16) are true. Moreover, we have (2.3.23) for $n = 1$ with

$$T^{\mu}_{1/1}(x) \equiv e : \bar{\psi}(x)\gamma^{\mu}\psi(x) : u(x). \tag{3.1.10}$$

We list below some obvious properties of the preceding expressions which are similar to the properties (2.1.14), (2.1.15) and (2.1.16):

$$U_{a,A}T^{\mu}_{1/1}(x)U^{-1}_{a,A} = \delta(A^{-1})^{\mu}{}_{\rho}T^{\rho}_{1/1}(\delta(A)\cdot x + a), \quad \forall A \in SL(2,\mathbb{C}),$$
$$U_{I_s}T^{\mu}_{1/1}(x)U^{-1}_{I_s} = (I_t)^{\mu}{}_{\rho}T^{\rho}_{1/1}(I_s \cdot x), \quad U_{I_t}T^{\mu}_{1/1}(x)U^{-1}_{I_t} = (I_t)^{\mu}{}_{\rho}T^{\rho}_{1/1}(I_t \cdot x), \tag{3.1.11}$$

$$\left[T^{\mu}_{1/1}(x), T^{\rho}_{1/1}(y)\right] = 0, \quad \left[T^{\mu}_{1/1}(x), T_1(y)\right] = 0, \quad \forall x, y \in \mathbb{R}^4 \quad s.t. \quad x \sim y \tag{3.1.12}$$

and

$$T^{\mu}_{1/1}(x)^{\dagger} = T^{\mu}_{1/1}(x). \tag{3.1.13}$$

To these one must add charge conjugation invariance:

$$U_C T^{\mu}_{1/1}(x)U^{-1}_C = T^{\mu}_{1/1}(x). \tag{3.1.14}$$

These properties can be easily deduced from the definitions of the various symmetry transformations and the explicit expression (3.1.10).

3.2 Gauge Invariance of Quantum Electrodynamics

In this Subsection we prove the following theorem.

Theorem 3.1 *Suppose that T_1 is given by (3.1.9) above. Then the distributions T_n can be constructed such that, beside the conditions of symmetry, $SL(2,\mathbb{C})$-covariance, causality and unitarity (2.1.1), (2.1.2), (2.1.4), (2.1.11), also verify covariance with respect to spatial and temporal inversions, invariance to charge conjugation and gauge invariance:*

$$U_{I_s}T_n(x_1,\ldots,x_n)U^{-1}_{I_s} = T_n(I_s \cdot x_1,\ldots,I_s \cdot x_n), \tag{3.2.1}$$

$$U_{I_t}T_n(x_1,\ldots,x_n)U^{-1}_{I_t} = \bar{T}_n(I_t \cdot x_1,\ldots,I_t \cdot x_n), \tag{3.2.2}$$

$$U_C T_n(x_1,\ldots,x_n)U^{-1}_C = T_n(x_1,\ldots,x_n), \tag{3.2.3}$$

$$d_Q T_n(x_1,\ldots,x_n) = i\sum_{l=1}^{n}\frac{\partial}{\partial x^{\mu}_l}T^{\mu}_{n/l}(x_1,\ldots,x_n), \quad \forall n \in \mathbb{N}^* \tag{3.2.4}$$

where $T^{\mu}_{n/l}$, $\quad l = 1,\ldots,n$ are some Wick monomials.

Proof:

(i) The main trick is to formulate carefully the **induction hypothesis**. We suppose that we have constructed the chronological products $T_p(x_1,\cdots,x_p)$, $\quad p = 1,\ldots,n-1$ having the following properties: (2.1.1), (2.1.4) and (2.1.11) for $p \leq n-1$, (2.1.4) and (2.1.5) for $|X_1| + |X_2| \leq n-1$ and

$$gh_{\pm}(T_p) = gh_{\pm}(\bar{T}_p) = 0 \tag{3.2.5}$$

for $p \leq n-1$.

We also suppose that we have constructed the Wick polynomials $T_{p/l}(x_1, \cdots, x_p), \quad l = 1, \ldots, p$ for $p = 1, \ldots, n-1$ such that we have properties analogue to (2.1.1), (2.1.4) and (2.1.11). We use a convention similar to (2.1.3): if $X = \{1, \ldots, p\}$ we denote $T_l^\mu(X) \equiv T_{p/l}^\mu(x_1, \ldots, x_p), \quad l \leq p$ and we assume that:

$$T(\emptyset) \equiv \mathbf{1}, \quad T_l^\mu(\emptyset) \equiv 0, \quad T_l^\mu(X) \equiv 0, \quad \text{for} \quad l \notin X. \tag{3.2.6}$$

Then the induction hypothesis is supplemented as follows.

- Symmetry:
$$T_{p/\pi(l)}(x_{\pi(1)}, \cdots x_{\pi(p)}) = T_{p/l}(x_1, \cdots x_p), \quad \forall \pi \in \mathcal{P}_p. \tag{3.2.7}$$
for $p = 1, \ldots, n-1$;

- Covariance with respect to $SL(2, \mathbb{C})$, spatial and temporal inversions:
$$U_{a,A} T_{p/l}^\mu(x_1, \cdots, x_p) U_{a,A}^{-1} = \delta(A^{-1})^\mu{}_\rho T_p^\rho(\delta(A) \cdot x_1 + a, \cdots, \delta(A) \cdot x_p + a), \tag{3.2.8}$$
$$U_{I_s} T_{p/l}^\mu(x_1, \cdots, x_p) U_{I_s}^{-1} = (I_t)^\mu{}_\rho T_p^\rho(I_s \cdot x_1, \cdots, I_s \cdot x_p), \tag{3.2.9}$$
$$U_{I_t} T_{p/l}^\mu(x_1, \cdots, x_p) U_{I_t}^{-1} = (I_t)^\mu{}_\rho T_p^\rho(I_t \cdot x_1, \cdots, I_t \cdot x_p), \tag{3.2.10}$$
for $p = 1, \ldots, n-1$;

- Charge conjugation:
$$U_C T_{p/l}^\mu(x_1, \cdots, x_p) U_C^{-1} = T_{p/l}^\mu(x_1, \cdots, x_p), \tag{3.2.11}$$
for $p = 1, \ldots, n-1$;

- Causality
$$T_l^\mu(X_1 X_2) = T_l^\mu(X_1) T(X_2) + T(X_1) T_l^\mu(X_2) \quad \forall X_1 \geq X_2 \tag{3.2.12}$$
and
$$[T_{l_1}^{\mu_1}(X_1), T_{l_2}^{\mu_2}(X_2)] = 0, \quad [T_l^\mu(X_1), T(X_2)] = 0 \quad \text{if} \quad X_1 \sim X_2 \tag{3.2.13}$$
for $|X_1| + |X_2| \leq n-1$.

- Unitarity; we introduce, in analogy to (2.1.7):
$$(-1)^{|X|} \bar{T}_l^\mu(X) \equiv \sum_{r=1}^{|X|} (-1)^r \sum [T_l^\mu(X_1) T(X_2) \cdots T(X_r) + \cdots$$
$$+ T(X_1) \cdots T(X_{r-1}) T_l^\mu(X_r)] \tag{3.2.14}$$
where $X_1, \cdots, X_r$ is a partition of X and we use in an essential way the convention (3.2.6). We require
$$\bar{T}_l^\mu(X) = T_l^\mu(X)^\dagger, \quad \forall X \tag{3.2.15}$$
for $|X| \leq n-1$;

- Gauge invariance:
$$d_Q T(X) = i \sum_{l \in X} \frac{\partial}{\partial x_l^\mu} T_l^\mu(X) \tag{3.2.16}$$
for all $|X| \leq n-1$. The restriction $l \in X$ is not essential because of the convention (3.2.6).

- Ghost number:
$$gh_+(T^\mu_l(X)) = gh_+(\bar{T}^\mu_l(X)) = 1, \quad gh_-(T^\mu_l(X)) = gh_-(\bar{T}^\mu_l(X)) = 0 \tag{3.2.17}$$
for $|X| \leq n-1$.

(ii) We observe that the induction hypothesis is valid for $p = 1$ according to (i). We suppose that it is true for $p \leq n-1$ and prove it for $p = n$.

First we establish in analogy to (2.1.12) that we have:
$$\sum (-1)^{|X|} \left[T^\mu_l(X)\bar{T}(Y) + T(X)\bar{T}^\mu_l(Y)\right] = 0 = \sum (-1)^{|X|} \left[\bar{T}^\mu_l(X)T(Y) + \bar{T}(X)T^\mu_l(Y)\right]. \tag{3.2.18}$$
where the sum goes over all partitions X, Y with $|X| + |Y| \leq n-1$.

Also one has, similarly to (2.1.13):
$$\bar{T}^\mu_l(XY) = \bar{T}^\mu_l(Y)\bar{T}(X) + \bar{T}(Y)\bar{T}^\mu_l(X), \quad \forall X \geq Y. \tag{3.2.19}$$
and $|X| + |Y| \leq n-1$.

Finally, from (3.2.4) and the definitions of the antichronological products $T(X)$ and $T^\mu(X)$ we have
$$d_Q\bar{T}(X) = i \sum_{l \in X} \frac{\partial}{\partial x^\mu_l} \bar{T}^\mu_l(X), \tag{3.2.20}$$
for all $|X| \leq n-1$.

Now we can proceed in strict analogy with Subsection 2.2. The proof of the following items below goes in strict analogy to the proof of the similar statements from the previous Subsection and can be easily provided with minimal modifications.

1. One constructs from $T(X), \quad T^\mu_l(X), \quad |X| \leq n-1$ the expressions $\bar{T}(X), \quad \bar{T}^\mu_l(X), \quad |X| \leq n-1$ and proves the properties (2.1.13) + (3.2.19) for $|X| + |Y| \leq n-1$ and (2.1.12) + (3.2.18) for $|X| \leq n-1$.

2. Beside lemma 2.3 we have the following result:

3. **Lemma 3.2** *Let us defines the expressions:*
$$A'^\mu_{n/l}(x_1, \ldots, x_{n-1}; x_n) \equiv \sum{}' (-1)^{|Y|} \left[T^\mu_l(X)\bar{T}(Y) + T(X)\bar{T}^\mu_l(Y)\right] \tag{3.2.21}$$
$$R'^\mu_{n/l}(x_1, \ldots, x_{n-1}; x_n) \equiv \sum{}' (-1)^{|Y|} \left[\bar{T}^\mu_l(X)T(Y) + \bar{T}(X)T^\mu_l(Y)\right] \tag{3.2.22}$$
where the sum $\sum'$ *goes over the partitions* $X \cup Y = \{1, \ldots, n\}, \quad X \cap Y = \emptyset, \quad Y \neq \emptyset, \quad x_n \in X$.
Now, let us suppose that we have a partition $P \cup Q = \{1, \ldots, n-1\}, \quad P \cap Q = \emptyset, \quad P \neq \emptyset$.
Then:
If $Qj \geq P$ *one has:*
$$A'^\mu_{n/l}(x_1, \ldots, x_{n-1}; x_n) = -\left[T^\mu_l(Qn)T(P) + T(Qn)T^\mu_l(P)\right] \tag{3.2.23}$$
and if $Qj \leq P$ *one has:*
$$R'^\mu_{n/l}(x_1, \ldots, x_{n-1}; x_n) = -\left[T^\mu_l(P)T(Qn) + T(P)T^\mu_l(Qn)\right]. \tag{3.2.24}$$

The proof is similar to the proof of lemma 2.3 if one uses the causality properties (3.2.12) and (3.2.19).

4. Beside corollary 2.4 we have:

 Corollary 3.3 *The expression*

$$D_n^\mu(x_1,\ldots,x_{n-1};x_n) \equiv A'_{n/l}(x_1,\ldots,x_{n-1};x_n) - R'_{n/l}(x_1,\ldots,x_{n-1};x_n). \tag{3.2.25}$$

 have causal support i.e. $supp(D_n^\mu(x_1,\ldots,x_{n-1};x_n)) \subset \Gamma^+(x_n) \cup \Gamma^-(x_n)$.

 The proof goes exactly as the proof of the Corollary 2.4.

5. In lemma 2.5 we must use the fact that $dim(T_1) = 4$; we also have the generalization:

 Lemma 3.4 *The distribution* $D_n^\mu(x_1,\ldots,x_{n-1};x_n)$ *can be written as a sum*

$$D_n^\mu(x_1,\ldots,x_{n-1};x_n) = \sum_i d_i^Q(x_1,\ldots,x_{n-1};x_n) W_i^Q(x_1,\ldots,x_{n-1};x_n) \tag{3.2.26}$$

 with $W_i^Q(x_1,\ldots,x_{n-1};x_n)$ *are linearly independent Wick monomials and* $d_i^Q(x_1,\ldots,x_{n-1};x_n)$ *numerical distributions with causal support i.e* $supp(d_i^Q(x_1,\ldots,x_{n-1};x_n)) \subset \Gamma^+(x_n) \cup \Gamma^-(x_n)$. *Moreover, the set of Wick monomials appearing in the preceding formula can be obtained from the expression* $T_1(x_1)\cdots T_l^\mu(x_i)\cdots T_n(x_n)$ *by taking all possible Wick contractions involving at least one field from* $T_l^\mu(x_i)$, *eliminating the monomials for for which the corresponding numerical distributions are factorizable and keeping a linearly independent set.*

 Finally, the following limitations are valid:

$$\omega(d_i^Q) + deg(W_i^Q) \leq deg(T_1) = 4, \quad \forall i. \tag{3.2.27}$$

6. **Corollary 3.5** *There exists a* $SL(2,\mathbb{C})$*-covariant causal splitting:*

$$D_{n/l}^\mu(x_1,\ldots,x_{n-1};x_n) = A_{n/l}^\mu(x_1,\ldots,x_{n-1};x_n) - R_{n/l}^\mu(x_1,\ldots,x_{n-1};x_{n/l}) \tag{3.2.28}$$

 with $supp(A_{n/l}^\mu(x_1,\ldots,x_{n-1};x_n)) \subset \Gamma^+(x_n)$ *and* $supp(R_{n/l}^\mu(x_1,\ldots,x_{n-1};x_n)) \subset \Gamma^-(x_n)$ *for all* $l = 1,\ldots,n$.

 The proof goes as in the case of the distribution D_n if one notices that the distributions $d_i^Q(x_1,\ldots,x_{n-1};x_n)$ defined above are $SL(2,\mathbb{C})$-covariant. For this reason $A_n(R_n)$ are called *advanced* (resp. *retarded*) products.

7. Beside lemma 2.9 we have

 Lemma 3.6 *The following relation is true*

$$D_{n/l}^\mu(x_1,\ldots,x_{n-1};x_n)^\dagger = (-1)^{n-1} D_{n/l}^\mu(x_1,\ldots,x_{n-1};x_n). \tag{3.2.29}$$

 In particular the causal splitting obtained above can be chosen such that

$$A_{n/l}^\mu(x_1,\ldots,x_{n-1};x_n)^\dagger = (-1)^{n-1} A_{n/l}^\mu(x_1,\ldots,x_{n-1};x_n). \tag{3.2.30}$$

So, performing the substitutions:

$$A^{\mu}_{n/l}(x_1,\dots,x_{n-1};x_n) \to \frac{1}{2}\left[A^{\mu}_{n/l}(x_1,\dots,x_{n-1};x_n)^{\dagger} + (-1)^{n-1}A^{\mu}_{n/l}(x_1,\dots,x_{n-1};x_n)\right]$$
$$R^{\mu}_{n/l}(x_1,\dots,x_{n-1};x_n) \to \frac{1}{2}\left[R^{\mu}_{n/l}(x_1,\dots,x_{n-1};x_n)^{\dagger} + (-1)^{n-1}R^{\mu}_{n/l}(x_1,\dots,x_{n-1};x_n)\right] \tag{3.2.31}$$

for all $l = 1,\dots,n$ we do not affect the relation from the preceding corollary and we obtain a causal splitting verifying the condition from the statement without spoiling the $SL(2,\mathbb{C})$-covariance. ■

8. Now we have again theorem 2.10 and also

Theorem 3.7 *Let us define*

$$T^{\mu}_{n/l}(x_1,\cdots,x_n) \equiv A^{\mu}_{n/l}(x_1,\cdots,x_{n-1};x_n) - A'^{\mu}_{n/l}(x_1,\cdots,x_{n-1};x_n)$$
$$\equiv R^{\mu}_{n/l}(x_1,\cdots,x_{n-1};x_n) - R'^{\mu}_{n/l}(x_1,\cdots,x_{n-1};x_n). \tag{3.2.32}$$

Then these expressions satisfy the Poincaré covariance, causality and unitarity conditions (3.2.8) (3.2.12) (3.2.13) and (3.2.15) for p = n. If we substitute

$$T^{\mu}_{n/l}(x_1,\cdots,x_n) \to \frac{1}{n!}\sum_{\pi} T^{\mu}_{n/\pi^{-1}(l)}(x_{\pi(1)},\cdots,x_{\pi(n)}) \tag{3.2.33}$$

where the sum runs over all permutations of the numbers $\{1,\dots,n\}$ *then we also have the symmetry axiom (3.2.7) for p = n.*

(iii) Now we investigate the possible obstruction to the extension of the identity (3.2.4) for $|X| = n$. We have the following results.

Proposition 3.8 *One can choose the expressions* $T(X), \quad T^{\mu}_{l}(X), \quad |X| = n$ *in such a way that they verify all the properties from the preceding theorem and, moreover, they are covariant with respect to spatial and temporal invariance and charge conjugation invariant.*

The proof for $T(X)$ is given in [38], ch. 4.4 and the proof for $T^{\mu}_{l}(X)$ is similar. Now follows a central result.

Proposition 3.9 *The following relation is valid:*

$$d_Q T(X) = i\sum_{l\in X}\frac{\partial}{\partial x^{\mu}_{l}}T^{\mu}_{l}(X) + P(X), \quad |X| = n \tag{3.2.34}$$

where $P(X) \equiv P_n(x_1,\dots,x_n)$ *is a Wick polynomial (called anomaly) of the following structure:*

$$P_n(x) = \sum_i p_i(\partial)\delta^{n-1}(x)W^{Q}_{i}(x); \tag{3.2.35}$$

here p_i *are polynomials in the derivatives with the maximal degree restricted by*

$$deg(p_i) + deg(W^{Q}_{i}) \leq 5. \tag{3.2.36}$$

Moreover, we have the following properties:

1. *Symmetry*
$$P_n(x_{\pi(1)},\ldots,x_{\pi(n)}) = P_n(x_1,\ldots,x_n) \tag{3.2.37}$$
for any permutation $\pi \in \mathcal{P}_n$.

2. $SL(2,\mathbb{C})$*-covariance:*
$$U_{a,A} P_n(x_1,\cdots,x_n) U_{a,A}^{-1} = P_n(\delta(A)\cdot x_1 + a,\cdots,\delta(A)\cdot x_n + a), \quad \forall (a,A) \in inSL(2,\mathbb{C}). \tag{3.2.38}$$

3. *Spatial inversion covariance:*
$$U_{I_s} P_n(x_1,\ldots,x_n) U_{I_s}^{-1} = P_n(I_s\cdot x_1,\ldots,I_s\cdot x_n). \tag{3.2.39}$$

4. *Temporal inversion covariance:*
$$U_{I_t} P_n(x_1,\ldots,x_n) U_{I_t}^{-1} = (-1)^n P_n(I_t\cdot x_1,\ldots,I_t\cdot x_n). \tag{3.2.40}$$

5. *Charge conjugation invariance:*
$$U_C P_n(x_1,\ldots,x_n) U_C^{-1} = P_n(x_1,\ldots,x_n). \tag{3.2.41}$$

6. *Unitarity:*
$$P_n^\dagger \equiv (-1)^n P_n. \tag{3.2.42}$$

7. *Ghost numbers restrictions:*
$$gh_+(P_n) = 1, \quad gh_-(P_n) = 0. \tag{3.2.43}$$

Proof: First we obtain from the lemmas 2.3 and 3.2 that:

$$\begin{aligned} d_Q A'_n(x_1,\ldots,x_{n-1};x_n) &= i\sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} A'^\mu_{n/l}(x_1,\ldots,x_n), \\ d_Q R'_n(x_1,\ldots,x_{n-1};x_n) &= i\sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} R'^\mu_{n/l}(x_1,\ldots,x_n) \end{aligned} \tag{3.2.44}$$

and by substraction we get:

$$d_Q D_n(x_1,\ldots,x_{n-1};x_n) = i\sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} D^\mu_{n/l}(x_1,\ldots,x_n), \tag{3.2.45}$$

We substitute here the causal decompositions (2.2.20) and (3.2.28) in the preceding relation and we get:

$$\begin{aligned} d_Q A_n(x_1,\ldots,x_{n-1};x_n) - i\sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} A^\mu_{n/l}(x_1,\ldots,x_n) &= \\ d_Q R_n(x_1,\ldots,x_{n-1};x_n) - i\sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} R^\mu_{n/l}(x_1,\ldots,x_n). \end{aligned} \tag{3.2.46}$$

Now we can reasons as in lemma 2.7 - see formula (2.2.14) : the left hand side has support in $\Gamma^+(x_n)$ and the right hand side in $\Gamma^-(x_n)$ so the common value, denoted by P_n should have the support in $\Gamma^+(x_n) \cap \Gamma^-(x_n) = \{x_1 = \cdots = x_n\}$. This means that we have:

$$d_Q A_n(x_1, \ldots, x_{n-1}; x_n) - i \sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} A^\mu_{n/l}(x_1, \ldots, x_{n-1}; x_n) = P'_n(x_1, \ldots, x_{n-1}; x_n) \qquad (3.2.47)$$

where P'_n has the structure (3.2.35) from the statement. We now have immediately the relation (3.2.34) from the statement where P_n has the structure (3.2.35). The limitation (3.2.36) follows rather easily from the lemmas 2.5 and 3.4. The same is true for the ghost number restriction (3.2.43). The restrictions (3.2.37), (3.2.39) and (3.2.41) follow from the similar properties of the products $T(X)$ and $T^\mu_l(X)$ from the preceding proposition.

For the temporal inversion and unitarity we proceed as follows. First, we apply the BRST operator d_Q to the relation (2.1.12) with $Z \equiv XY$ of cardinal n. If we use the induction hypothesis (3.2.4) and (3.2.20) + (3.2.34) we get:

$$d_Q \bar{T}(X) = i \sum_{l \in X} \frac{\partial}{\partial x_l^\mu} \bar{T}^\mu_l(X) + \bar{P}(X), \quad |X| = n \qquad (3.2.48)$$

where

$$\bar{P}_n \equiv (-1)^{n-1} P_n. \qquad (3.2.49)$$

Now we apply $U_{I_t} \cdots U_{I_t}^{-1}$ to relation (3.2.34). One can prove that

$$d_Q U_{I_t} = -U_{I_t} d_Q. \qquad (3.2.50)$$

because the temporal inversion is anti-unitary. We compare the result with the preceding relation. The relation (3.2.40) follows.

Finally, to get the relation (3.2.42), we first prove the formula

$$(d_Q W)^\dagger = -d_Q(W^\dagger) \qquad (3.2.51)$$

for any Wick monomial, we apply the conjugation † to the relation (3.2.4) and compare to the relation (3.2.48). ∇

(iv) There are a lot of restrictions on the anomaly P_n and we will be able to prove here that it can be chosen to be equal to 0. First, from the restrictions (3.2.36)) and the $SL(2, \mathbb{C})$-covariance (3.2.38) we obtain that

$$P_n = \sum_{i=1}^{12} \mathcal{P}_i \qquad (3.2.52)$$

where the list of the polynomials in the right hand side is:

$$\mathcal{P}_1 \equiv \sum_{ijl} :\bar{\psi}(x_i) p^\rho_{ijl}(x) \psi(x_j): \partial_\rho u(x_l), \quad \mathcal{P}_2 \equiv \sum_{ijl} : \left[\partial_\rho \bar{\psi}(x_i)\right] q^\rho_{ijl}(x) \psi(x_j) : u(x_l),$$

$$\mathcal{P}_3 \equiv \sum_{ijl} :\bar{\psi}(x_i) r^\rho_{ijl}(x) \left[\partial_\rho \psi(x_j)\right] : u(x_l), \quad \mathcal{P}_4 \equiv \sum_{ijl} :\bar{\psi}(x_i) p_{ijl}(x) \psi(x_j) : u(x_l),$$

$$\mathcal{P}_5 \equiv \sum_{ijkl} :\bar{\psi}(x_i) p^\rho_{ijkl}(x) \psi(x_j) : A_\rho(x_k) u(x_l), \quad \mathcal{P}_6 \equiv \sum_{l} p_l(x) u(x_l),$$

$$\mathcal{P}_7 \equiv \sum_{kl} p^{\rho}_{kl}(x) A_\rho(x_k) u(x_l), \quad \mathcal{P}_8 \equiv \sum_{kl} p^{\rho\lambda}_{kl}(x) A_\rho(x_k) \partial_\lambda u(x_l),$$
$$\mathcal{P}_9 \equiv \sum_{klm} p^{\rho\lambda}_{klm}(x) : A_\rho(x_k) A_\rho(x_m) : u(x_l),$$
$$\mathcal{P}_{10} \equiv \sum_{klm} p^{\rho\lambda\zeta}_{klm}(x) : A_\rho(x_k) A_\rho(x_m) : \partial_\zeta u(x_l),$$
$$\mathcal{P}_{11} \equiv \sum_{klmr} p^{\rho\lambda\zeta}_{klmr}(x) : A_\rho(x_k) A_\rho(x_m) A_\zeta(x_r) : u(x_l),$$
$$\mathcal{P}_{12} \equiv \sum_{klmr} p^{\rho\lambda\zeta\tau}_{klmr}(x) : A_\rho(x_k) A_\rho(x_m) A_\zeta(x_r) : \partial_\tau u(x_l) \tag{3.2.53}$$

where the expressions $p^{\cdots}_{\cdots}$ are numerical distributions which are $SL(2,\mathbb{C})$-covariant and are also restricted by the following degree conditions:

$$deg(p^{\rho}_{ijl}), \quad deg(q^{\rho}_{ijl}), \quad deg(r^{\rho}_{ijl}), \quad deg(p^{\rho}_{ijkl}), \quad deg(p^{\rho\lambda\zeta\tau}_{klmr}) \leq 0,$$
$$deg(p_{ijl}), \quad deg(p^{\rho\lambda\zeta}_{klm}), \quad deg(p^{\rho\lambda\zeta}_{klmr}) \leq 1,$$
$$deg(p^{\rho\lambda}_{kl}), \quad deg(p^{\rho\lambda}_{klm}), \quad \leq 2,$$
$$deg(p^{\rho}_{kl}) \leq 3, \quad deg(p_l) \leq 4. \tag{3.2.54}$$

It is obvious that all these polynomials also verify individually all the restrictions from the preceding proposition. In particular, charge conjugation invariance gives immediately:

$$\mathcal{P}_i = 0, \quad i = 6, 9, 10. \tag{3.2.55}$$

We analyse now the other cases. The basic idea is to perform obvious "integrations by parts" and exhibit the polynomials as follows:

$$\mathcal{P}_i(x) = d_Q N(x) + i \sum_{l=1}^{n} \frac{\partial}{\partial x^\rho_l} N^\rho_l(x) + +\delta^{n-1}(x) W(x_n) \tag{3.2.56}$$

for some Wick monomial $W(x)$. The first two terms can be eliminated by a suitable redefinition of the expressions T_n and $T^\mu_{n/l}$ and it remains to prove that the last is zero because of invariance with respect to some discrete symmetry considered above. We give below the details.

1) In this case we have the following generic form of the numerical distribution:

$$p^{\rho}_{ijl}(x) = \sum_{ijl} \left(K_{ijl} \gamma^\rho + K'_{ijl} \gamma^\rho \gamma_5 \right) \delta^{n-1}(x). \tag{3.2.57}$$

This means that we have:

$$\mathcal{P}_1 = \delta^{n-1}(x) \quad \left[K : \bar{\psi}(x_n) \gamma^\rho \psi(x_n) : + K' : \bar{\psi}(x_n) \gamma^\rho \gamma_5 \psi(x_n) : \right] \partial_\rho u(x_n). \tag{3.2.58}$$

But one proves by simple computations that the first term is be zero because spatial inversion covariance and the last term is zero because charge conjugation covariance.

2), 3) 4) In the first two cases the structure of the numerical distributions q and r is similar to the structure (3.2.57) above and it follows that the sum of these two contributions is of the form:

$$\mathcal{P}_0(x) = \delta^{n-1}(x) \quad \left[K : \bar{\psi}(x_n) \psi(x_n) : + K' : \bar{\psi}(x_n) \gamma_5 \psi(x_n) : \right] u(x_n). \tag{3.2.59}$$

In the case 4) we have the generic form

$$p_{ijl} = \sum_{ijl} \left(K_{ijl}\mathbf{1} + K'_{ijl}\gamma_5 + K_{ijlm}\gamma \cdot \partial_m + K'_{ijl}\gamma_5\gamma \cdot \partial_m \right) \delta^{n-1}(x). \tag{3.2.60}$$

By performing "integrations by parts" we get a formula of the type (3.2.56) with $N = 0$: and the Wick polynomial W of the following form:

$$\begin{aligned} W(x) = \{K_1 : [\partial_\rho\bar{\psi}(x)]\gamma^\rho\psi(x) : +K_2 : \bar{\psi}(x)\gamma^\rho[\partial_\rho\psi(x)] : \\ +K'_1 : [\partial_\rho\bar{\psi}(x)]\gamma^\rho\gamma_5\psi(x) : +K'_2 : \bar{\psi}(x)\gamma_5\gamma^\rho[\partial_\rho\psi(x)] :\}u(x) \\ +[K_3 : \bar{\psi}(x)\gamma^\rho\psi(x) : +K_4 : \bar{\psi}(x)\gamma^\rho\psi(x) :]\partial_\rho u(x). \end{aligned} \tag{3.2.61}$$

If we use Dirac equation (3.1.3) we can rewrite W as follows:

$$W(x) = d_Q N(x) + \left[K : \bar{\psi}(x)\psi(x) : +K' : \bar{\psi}(x)\gamma_5\psi(x) :\right] u(x) \tag{3.2.62}$$

where:

$$N(x) = -i\left[K_3 : \bar{\psi}(x)\gamma^\rho\psi(x) : +K_4 : \bar{\psi}(x)\gamma_5\psi(x)\right] A_\rho(x). \tag{3.2.63}$$

So, we have in the end we have (3.2.56) with W having the same structure as $\mathcal{P}_0$. It is not very difficult to se that we have:

$$N^\dagger = (-1)^{n-1}N, \quad (N_l^\rho)^\dagger = (-1)^{n-1}N_l^\rho \tag{3.2.64}$$

so if we make the redefinitions:

$$\begin{aligned} T_n \rightarrow T_n + N, \quad \bar{T}_n \rightarrow \bar{T}_n + (-1)^{n-1}N, \\ T^\rho_{n/l} \rightarrow T^\rho_{n/l} + N, \quad \bar{T}^\rho_{n/l} \rightarrow \bar{T}^\rho_{n/l} + (-1)^{n-1}N_l^\rho \end{aligned} \tag{3.2.65}$$

we do not affect the properties of the distributions $T(X), \quad T^\rho_L(X), \quad |X| = n$ and we can take $\mathcal{P}_4$ of the form $\mathcal{P}_0$. In other words, by suitable redefining the chronological products, we can arrange such that the sum of the contributions 2), 3) and 4) is of the form $\mathcal{P}_0$. Now it follows by elementary computations that charge conjugation invariance imposes $\mathcal{P}_0 = 0$.

5) In this case, the numerical distributions $p_{\cdots}$ have the same structure as in the case 1) so we end up with

$$\mathcal{P}_5(x) = \delta^{n-1}(x) \quad \left[K : \bar{\psi}(x_n\gamma^\rho\psi(x_n) : +K' : \bar{\psi}(x_n\gamma^\rho\gamma_5\psi(x_n) :\right] u(x_n)A_\rho(x_n)u(x_n). \tag{3.2.66}$$

The first contribution is zero because of the charge conjugation invariance and the second contribution is zero because the spatial inversion covariance.

7) In this case we have

$$p^\mu(x) = \sum_i \left(C_i\partial_i^\mu + \sum_{ijk} D_{ijk}\partial_i^\mu\partial_j \cdot \partial_k + \varepsilon^{\mu\nu\rho\lambda}D'_{ijk}\partial_{i\nu}\partial_{j\rho}\partial_{k\lambda} \right) \delta^{n-1}(x). \tag{3.2.67}$$

We integrate by parts and end up with (3.2.56) with:

$$W(x) = C_1\partial^\mu A_\mu u + C_2\partial^\mu u A_\mu + +C_3(\partial_\nu A_\mu)(\partial^\mu\partial^\nu u) + +C_4(\partial^\mu\partial^\nu A_\mu)(\partial_\nu u). \tag{3.2.68}$$

(The equations of motion (2.3.1) have been used). This term must be zero because of the covariance with respect to spatial inversion.

8) In this case we have

$$p^{\mu\rho} = \left(C_0 g^{\mu\rho} + \sum_{ij} C_{ij} \partial_i^\mu \partial_j^\rho + \sum_i C_i \partial_i^2 g^{\mu\rho} + \varepsilon^{\mu\nu\rho\lambda} \sum_{ij} C'_{ij} \partial_{i\nu} \partial_{j\lambda} \right) \delta^{n-1}(x). \tag{3.2.69}$$

We integrate by parts and obtain (3.2.56) with $W = 0$ and

$$N(x) = -\frac{i}{2} \delta^{n-1}(x) \left[C_1 : A_\mu(x_n) A^\mu(x_n) : + C_2 : \partial_\rho A_\mu(x_n) \partial^\rho A^\mu(x_n) : \right]. \tag{3.2.70}$$

Again, we can make $\mathcal{P}_8 = 0$ by suitable redefinitions of the chronological products.

11) In this case we have

$$p^{\mu\nu\rho} = \sum_i \left(C_i g^{\mu\nu} \partial_i^\rho + D_i g^{\mu\rho} \partial_i^\nu + E_i g^{\rho\nu} \partial_i^\mu + \varepsilon^{\mu\nu\rho\lambda} F_i \partial_{i\lambda} \right) \delta^{n-1}(x). \tag{3.2.71}$$

We integrate by parts and end up with (3.2.56) with

$$W = K_1 : A_\mu A^\mu \partial^\nu A_\nu : u + K_2 : A_\mu A^\mu A_\nu : \partial^\nu u. \tag{3.2.72}$$

This contribution is zero because of the covariance with respect to the spatial inversion.

12) In this case we have

$$p^{\mu\nu\rho\lambda} = \left(K_1 g^{\mu\nu} g^{\rho\lambda} + K_2 g^{\mu\rho} g^{\nu\lambda} + K_3 g^{\mu\lambda} g^{\rho\nu} + K_4 \varepsilon^{\mu\nu\rho\lambda} \right) \delta^{n-1}(x). \tag{3.2.73}$$

so

$$\mathcal{P}_{12}(x) = C \delta^{n-1}(x) : A_\mu(x_n) A^\mu(x_n) A_\rho(x_n) : \partial^\rho u(x_n) \tag{3.2.74}$$

which is zero because of the covariance with respect to spatial inversions.

In conclusion, we can make in (3.2.4) $P_n = 0$. Next we fix the invariance properties with respect to the discrete symmetries as in [38] ch. 4.4 and the chronological products $T(X), \quad |X| = n$ have all the required properties. This finishes the proof. ■

We now determine the non-unicity of the chronological products $T(X)$. We have:

Proposition 3.10 *Suppose that $T(X)$ and $T'(X)$ are two solutions of the renormalization theory for quantum electrodynamics, verifying gauge invariance in the sense of the preceding theorem and the power counting condition (2.2.27). Then we have*

$$T(X) - T'(X) = d_Q N(X) + i \sum_{l \in X} \frac{\partial}{\partial x_l^\mu} N_l^\mu(X) + C \delta^{n-1}(x) : \bar\psi(x_n) \gamma^\rho \psi(x_n) : A_\mu(x_n) \tag{3.2.75}$$

with $i^n C \in \mathbb{R}$. In particular, we can absorb the last term in the interaction Lagrangian by redefining the coupling constant up to order n.

Proof: From the gauge invariance condition, the expression $F(X) \equiv T(X) - T'(X)$ verifies:

$$d_Q F(X) = i \sum_{l \in X} \frac{\partial}{\partial x_l^\mu} F_l^\mu(X) \tag{3.2.76}$$

for some Wick polynomials $F_l^{\mu}(X)$. Now we have from lemma 2.5

$$F(X) = \sum_i f_i(x) W_i(x) \tag{3.2.77}$$

with the numerical distributions of the form

$$f_i(x) = p_i(\partial)\delta^{n-1}(x) \tag{3.2.78}$$

where p_i are polynomials verifying the restrictions

$$deg(p_i) + deg(W_i) \leq deg(T_1) = 4, \quad \forall i. \tag{3.2.79}$$

We also have all the properties of symmetry, covariance with respect to $SL(2,\mathbb{C})$, spatial and temporal inversions and charge conjugation invariance. We list all polynomials fulfilling these requirements and we obtain the result. ■

Remark 3.11 *The possibility of redefining the charge such that the last term in (3.2.75) disappears is named charge renormalization. We see that, in a rigorous version of the renormalization theory there is no mass renormalization, because one starts from the very beginning in a definite Fock space where the Dirac particle has a fixed mass m.*

Remark 3.12 *It is plausible that some descent equations of the type:*

$$d_Q T^{\mu_1,\dots,\mu_k}_{l_1,\dots,l_k} = i \sum_{l_{k+1} \neq l_1,\dots,l_k} \frac{\partial}{\partial x^{\mu_{k+1}}_{k+1}} T^{\mu_1,\dots,\mu_{k+1}}_{l_1,\dots,l_{k+1}}, \quad k \leq n \tag{3.2.80}$$

can be established by refining the induction hypothesis. Such formulæ could be useful in more complicated theories.

We close this Section with a comparison between our proof and the proof appearing in [38], ch. 4.6. In this reference one works in a quantization formalism for the electromagnetic field without ghosts. One can prove, also by induction, a more precise formula for the chronological products:

$$T_n(x_1,\dots,x_n) \sum_{I,J,K} : \prod_{i\in I} \bar{\psi}(x_i) t^{\mu_K}_{I,J,K}(X) \prod_{j\in J} \psi(x_j) :: \prod_{k\in K} A_{\mu_k}(x_k) : \tag{3.2.81}$$

where: a) the sum runs over all distinct triplets $I,J,K \subset \{1,\dots,n\}$ verifying $|I| = |J|$; b) by μ_K we mean the set $\{\mu_k\}_{k\in K}$; c) the expression $t^{\mu_K}_{I,J,K}$ are numerical distributions (in fact, they take values in the matrix space $M_{\mathbb{C}}(4,4)^{\otimes|I|}$.)

The possibility of having non-zero intersection between the sets I, J and K is permitted. The condition of gauge invariance (3.2.4) can be translated into conditions on these numerical distributions $t^{\mu_K}_{I,J,K}$ which are the famous Ward-Takahashi identities. They have a rather complicated form precisely because of the possible non-void intersections mentioned above. In fact, the proof from [38] relies on the following identities for which we did not found an elementary proof. Formula (4.6.36) from this reference is a particular case of the formulæ below.

$$\begin{aligned}
t^{\rho\mu_K}_{iI,J,iK}(X) &= e\sum_{j\notin I} \left[\gamma^\rho S^F_m(x_i - x_j) \otimes \cdots \otimes \mathbf{1}\right] t^{\mu_K}_{jI,J,K}(X\ x_i), \quad \forall i \notin I, |J| < n,\\
t^{\mu_K\rho}_{I,Jj,Kj}(X) &= e\sum_{i\notin J} t^{\mu_K}_{jI,JK,}(X\ x_j) \left[\mathbf{1} \otimes \cdots \otimes S^F_m(x_i - x_j)\gamma^\rho\right], \quad \forall j \notin J, |I| < n,\\
t^{\mu_K}_{iI,iJ,K}(X) &= e\sum_{l\notin K} D^F_0(x_i - x_l) t^{\mu_K\rho}_{I,J,Kl}(X\ x_l) \otimes \gamma_\rho \quad \forall i \notin I\cup J, |I| < n.
\end{aligned} \tag{3.2.82}$$

4 Yang-Mills Theories

In this Section we prove the same result given above for the case of the Standard Model (SM).

4.1 The Fock Space of the Bosons

We give some basic facts about the quantization of a spin 1 Boson of mass $m > 0$. One can proceed in a rather close analogy to the case of the photon; for more details see [26] and references quoted there. Let us denote the Hilbert space of the Boson by H_m; it carries the unitary representation of the orthochronous Poincaré group $\mathsf{H}^{[m,1]}$ (see [43]).

The Hilbert space of the multi-Boson system should be, as before, the associated symmetric Fock space $\mathcal{F}_m \equiv \mathcal{F}^+(\mathrm{H}_m)$. We construct this Fock space as before in the spirit of algebraic quantum field theory. One considers the Hilbert space $\mathcal{H}^{gh}$ generated by applying on the vacuum Φ_0 the free fields $A^\mu(x), \quad u(x), \quad \tilde{u}(x), \quad \Phi(x)$ which are completely characterize by the following properties:

- Equation of motion:

$$(\square + m^2)A^\mu(x), \quad (\square + m^2)u(x) = 0, \quad (\square + m^2)\tilde{u}(x) = 0, \quad (\square + m^2)\Phi(x) = 0. \tag{4.1.1}$$

- Canonical (anti)commutation relations:

$$\begin{gathered}
[A^\mu(x), A^\rho(y)] = -g^{\mu\rho} D_m(x-y) \times \mathbf{1}, \\
[A^\mu(x), u(y)] = 0, \quad [A^\mu(x), \tilde{u}(y)] = 0, \quad [A^\mu(x), \Phi(y)] = 0, \\
\{u(x), u(y)\} = 0, \quad \{\tilde{u}(x), \tilde{u}(y)\} = 0, \quad \{u(x), \tilde{u}(y)\} = D_m(x-y) \times \mathbf{1}, \\
\{\Phi(x), \Phi(y)\} = D_m(x-y) \times \mathbf{1}, \quad [\Phi(x), u(y)] = 0.
\end{gathered} \tag{4.1.2}$$

- $SL(2,\mathbb{C})$-covariance. We keep the corresponding relations (2.3.4) from Subsection 2.3 and we add:

$$U_{a,A}\Phi(x)U_{a,A}^{-1} = \Phi(\delta(A)\cdot x + a), \tag{4.1.3}$$

- PCT covariance.

$$\begin{gathered}
U_{PCT}A_\mu(x)U_{PCT}^{-1} = -A_\mu(-x), \quad U_{PCT}\Phi(x)U_{PCT}^{-1} = \Phi(-x) \\
U_{PCT}u(x)U_{PCT}^{-1} = -u(-x), \quad U_{PCT}\tilde{u}(x)U_{PCT}^{-1} = -\tilde{u}(-x), \\
U_{PCT}\Phi_0 = \Phi_0.
\end{gathered} \tag{4.1.4}$$

Remark 4.1 *Although we could give the expressions for $U_{I_s}, \quad U_{I_t}$ and U_C separately, we prefer to give only the expression of the PCT transform because the interaction Lagrangian of the standard model is not invariant with respect to these three operations but it is PCT-covariant.*

We give as before in $\mathcal{H}^{gh}$ the sesqui-linear form $< \cdot,\cdot >$ which is completely characterize by requiring beside (2.3.9):

$$\Phi(x)^\dagger = \Phi(x). \tag{4.1.5}$$

Now, the expression of the supercharge gets an extra term:

$$Q = \int_{\mathbb{R}^3} d^3x \, [\partial^\mu A_\mu(x) + m\Phi(x)] \overleftrightarrow{\partial_0} u(x) \tag{4.1.6}$$

and one can see that (2.3.11) stays true, and we must modify (2.3.12):

$$[Q, A_\mu] = i\partial_\mu u, \quad \{Q, u\} = 0, \{Q, \tilde{u}\} = -i(\partial_\mu A^\mu + m\Phi), \quad [Q, \Phi] = imu \tag{4.1.7}$$

We still have

$$Q^2 = 0 \Longrightarrow Im(Q) \subset Ker(Q) \tag{4.1.8}$$

and also

$$U_{a,A} Q = Q U_{a,A}, \quad U_{PCT} Q = Q U_{PCT}. \tag{4.1.9}$$

Finally:

Theorem 4.2 *The sesqui-linear form $< \cdot, \cdot >$ factorizes to a well-defined scalar product on the completion of the factor space $Ker(Q)/Im(Q)$. Then there exists the following Hilbert spaces isomorphism:*

$$\overline{Ker(Q)/Im(Q)} \simeq \mathcal{F}_m; \tag{4.1.10}$$

The representation of the Poincaré group and the PCT operator are factorizing to $Ker(Q)/Im(Q)$ and are producing unitary operators (resp. an anti-unitary operator).

If $\mathcal{W}$ the linear space of all Wick monomials acting in the Fock space $\mathcal{H}^{gh}$ containing the fields $A_\mu(x),\ u(x),\ \tilde{u}(x)$ and $\Phi(x)$ then the expression of the BRST operator is determined by

$$d_Q u = 0, \quad d_Q \tilde{u} = -i(\partial^\mu A_\mu + m\Phi), \quad d_Q A_\mu = i\partial_\mu u, \quad d_Q \Phi = imu. \tag{4.1.11}$$

and, as a consequence (2.3.21) stays true. Again we notice that this expression is a linearized form of the usual BRST transform.

If one adds matter fields we proceed as before. In particular, this will mean that the BRST operator acts trivially on the matter fields.

Now we can define the Yang-Mills field. We must consider the case when we have r fields of spin 1 and some of them will have zero mass and the others will be considered of non-zero mass. Apparently, we need the scalar ghosts only in the last case. However it can be shown that with this assumption, there are no non-trivial models. To avoid this situation, we make the following amendment. All the fields considered above will carry an additional index $a = 1, \ldots, r$ i.e. we have the following set of fields: $A_{a\mu},\ u_a,\ \tilde{u}_a,\ \Phi_a \quad a = 1, \ldots, r$.

These fields verify the following equations of motion:

$$(\square + m_a^2) A_a^\mu(x) = 0, \quad (\square + m_a^2) u_a(x) = 0, \quad (\square + m_a^2) \tilde{u}_a(x) = 0, \quad (\square + (m_a^H)^2) \Phi_a(x) = 0 \tag{4.1.12}$$

for all $a = 1, \ldots, r$ and we suppose that if $m_a \neq 0$, then we have $m_a^H = m_a$; However, if some Boson field $m_a = 0$, we do not take the corresponding scalar ghosts of zero-mass, but we consider them as physical fields (called *Higgs fields*) of arbitrary mass $m_a^H \geq 0$.

The rest of the formalism stays unchanged. The canonical (anti)commutation relations are:

$$\begin{gathered} [A_{a\mu}(x), A_{b\nu}(y)] = -\delta_{ab} g_{\mu\nu} D_{m_a}(x-y) \times \mathbf{1}, \\ \{u_a(x), \tilde{u}_b(y)\} = \delta_{ab} D_{m_a}(x-y) \times \mathbf{1}, \quad [\Phi_a(x), \Phi_b(y)] = \delta_{ab} D_{m_a}(x-y) \times \mathbf{1}; \end{gathered} \tag{4.1.13}$$

and all other (anti)commutators are null. The supercharge is given by

$$Q = \sum_{a=1}^{r} \int_{\mathbb{R}^3} d^3x \, [\partial^\mu A_{a\mu}(x) + m_a \Phi_a(x)] \overset{\leftrightarrow}{\partial_0} u_a(x) \tag{4.1.14}$$

and verifies all the expected properties.

The Krein operator is determined by:

$$A_{a\mu}(x)^\dagger = A_{a\mu}(x), \quad u_a(x)^\dagger = u_a(x), \quad \tilde{u}_a(x)^\dagger = -\tilde{u}_a(x), \quad \Phi_a(x)^\dagger = \Phi_a(x). \tag{4.1.15}$$

The ghost degree is defined in an obvious way and the expression of the BRST operator is similar to the previous one. In particular we have (see (2.3.20)):

$$d_Q u_a = 0, \quad d_Q \tilde{u}_a = -i(\partial_\mu A_a^\mu + m_a \Phi_a), \quad d_Q A_a^\mu = i\partial^\mu u_a, \quad d_Q \Phi_a = i m_a u_a, \quad \forall a = 1, \dots, r. \tag{4.1.16}$$

4.2 Matter Fields and the Interaction Lagrangian of the SM

In this case the matter field is a set of Dirac fields of mass $M_A, \quad A = 1, \dots, N$ denoted by $\psi_A(x)$.

As before, these fields should be characterized by $(A, B = 1, \dots, N)$:

- Equation of motion:
$$(i\gamma \cdot \partial + M_A)\psi_A(x) = 0. \tag{4.2.1}$$

- Canonical (anti)commutation relations:
$$\begin{gathered}[\psi_A(x), A_a^\mu(y)] = 0, \quad [\psi_A(x), u_a(y)] = 0, \quad [\psi_A(x), \tilde{u}_a(y)] = 0, \quad [\psi_A(x), \Phi_a(y)] = 0 \\ \{\psi_A(x), \psi_B(y)\} = 0, \quad \{\psi_A(x), \overline{\psi_B}(y)\} = \delta_{AB} S_{M_A}(x-y) \times \mathbf{1}.\end{gathered} \tag{4.2.2}$$

- Covariance properties with respect to the Poincaré group:
$$U_{a,A}\psi_A(x)U_{a,A}^{-1} = S(A^{-1})\psi_A(\delta(A)\cdot x + a). \tag{4.2.3}$$

- PCT-covariance:
$$U_{PCT}\psi_A(x)U_{I_s}^{-1} = \gamma_1\gamma_2\gamma_3\overline{\psi_A}(-x)^t. \tag{4.2.4}$$

The condition of gauge invariance remains the same:

$$d_Q T_n(x_1, \dots, x_n) = i\sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} T_l^\mu(x_1, \dots, x_n) \tag{4.2.5}$$

for some Wick polynomials $T_l^\mu(x_1, \dots, x_n), \quad l = 1, \dots, n$

One can prove [26] that this condition for $n = 1, 2$ determines quite drastically the interaction Lagrangian of degree 4:

$$\begin{aligned} T_1(x) \equiv f_{abc}\,[: A_{a\mu}(x)A_{b\nu}(x)\partial^\nu A_a^\mu(x) : - : A_a^\mu(x)u_b(x)\partial_\mu\tilde{u}_c(x) :]\,, \\ + f'_{abc}\,[: \Phi_a(x)\partial_\mu\Phi_b(x)A_c^\mu(x) : - m_b : \Phi_a(x)A_{b\mu}(x)A_c^\mu(x) : - m_b : \Phi_a(x)\tilde{u}_b(x)u_c(x) :] \\ + f^{"}_{abc} : \Phi_a(x)\Phi_b(x)\Phi_c(x) : + j_a^\mu(x)A_{a\mu}(x) + j_a(x)\Phi_a(x) \end{aligned} \tag{4.2.6}$$

where:

$$j_a^\mu(x) =: \overline{\psi_A}(x)(t_a)_{AB}\gamma^\mu\psi_B(x) : + : \overline{\psi_A}(x)(t'_a)_{AB}\gamma^\mu\gamma_5\psi_B(x) : \tag{4.2.7}$$

and

$$j_a(x) =: \overline{\psi_A}(x)(s_a)_{AB}\psi_B(x) : + : \overline{\psi_A}(x)(s'_a)_{AB}\gamma_5\psi_B(x) : \tag{4.2.8}$$

are the so-called *currents*. The conditions of $SL(2,\mathbb{C})$ and PCT-covariance of the interaction Lagrangian are easy to prove as well as the causality condition. The hermiticity conditions are equivalent to the fact that the complex $N \times N$ matrices $t_a, \quad t'_a, \quad s_a, \quad a = 1, \dots r$ are hermitian and $s'_a, \quad a = 1, \dots, r$ is anti-hermitian. We also have:

$$gh(T_1) = 0. \tag{4.2.9}$$

The constants f_{abc} are completely anti-symmetric and verify Jacobi identity so they generate a compact semi-simple Lie group quite naturally. There are other conditions on the rest of the constants as well, but because we do not need these properties in the proof of renormalizability, we refer to the literature [26], [27] and references quoted there.

Moreover, it can be proved that the condition of gauge invariance (4.2.5) is valid for $n = 1, 2$ and we can take the following expression for $T^{\mu}_{1/1}$:

$$\begin{aligned} T^{\mu}_{1/1} = f_{abc} \left(: u_a A_{b\nu} F^{\nu\mu}_c : -\frac{1}{2} : u_a u_b \partial^{\mu} \tilde{u}_c : \right) \\ + f'_{abc} \left(m_a : A^{\mu}_a \Phi_b u_c : + : \Phi_a \partial^{\mu} \Phi_b u_c : \right) . + u_a(x) j^{\mu}_a(x). \end{aligned} \tag{4.2.10}$$

This Wick polynomial verifies the following relations:

- $SL(2,\mathbb{C})$-covariance:

$$U_{a,A} T^{\mu}_{1/1}(x) U^{-1}_{a,A} = \delta(A^{-1})^{\mu}{}_{\rho} T^{\rho}_{1/1}(\delta(A) \cdot x + a), \quad \forall A \in SL(2,\mathbb{C}). \tag{4.2.11}$$

- PCT-covariance:

$$U_{PCT} T^{\mu}_{1/1}(x) U^{-1}_{PCT} = T^{\mu}_{1/1}(-x). \tag{4.2.12}$$

- Causality:

$$\left[T^{\mu}_{1/1}(x), T^{\rho}_{1/1}(y) \right] = 0, \quad \left[T^{\mu}_{1/1}(x), T_1(y) \right] = 0, \quad \forall x, y \in \mathbb{R}^4 \quad s.t. \quad x \sim y. \tag{4.2.13}$$

- Unitarity:

$$T^{\mu}_{1/1}(x)^{\dagger} = T^{\mu}_{1/1}(x). \tag{4.2.14}$$

4.3 Renormalization of the Standard Model

We prove that the standard model as defined in the previous Subsection is renormalizable, provided there are no anomalies in the third order. The proof will be extremely similar to the proof from Subsection 3.2 and we will indicate only the appropriate changes.

Theorem 4.3 *Suppose that T_1 is given by (4.2.6) above such that the condition of gauge invariance*

$$d_Q T_n(x_1,\dots,x_n) = i\sum_{l=1}^{n} \frac{\partial}{\partial x^\mu_l} T^\mu_{n/l}(x_1,\dots,x_n), \quad \forall n \in \mathbb{N}^* \tag{4.3.1}$$

(where $T^\mu_{n/l}$, $l = 1,\dots,n$ are some Wick monomials) is valid in the third order also. Then the distributions T_n can be constructed such that, beside the conditions of symmetry, $SL(2,\mathbb{C})$-covariance, causality and unitarity (2.1.1), (2.1.2), (2.1.4), (2.1.11), also gauge invariance (4.3.1) and PCT-covariance :

$$U_{PCT} T_n(x_1,\dots,x_n) U_{PCT}^{-1} = \bar{T}_n(-x_1,\dots,-x_n), \tag{4.3.2}$$

Proof:

(i) We formulate the induction hypothesis. The only differences that do appear are that instead of covariance with respect to spatial (and temporal) inversions and charge conjugation invariance we require only PCT-covariance:

$$\begin{aligned} U_{PCT} T_p(x_1,\cdots,x_p) U_{PCT}^{-1} &= \bar{T}_p(-x_1,\cdots,-x_p), \\ U_{PCT} T^\mu_{p/l}(x_1,\cdots,x_p) U_{PCT}^{-1} &= \bar{T}^\mu_p(-x_1,\cdots,-x_p) \end{aligned} \tag{4.3.3}$$

and we also weaken the ghost number hypothesis to:

$$gh(T_p) = 0, \quad gh(T^\mu_{p/l}) = 1 \tag{4.3.4}$$

for $p \leq n-1$.

We observe that the induction hypothesis is valid for $p = 1$ according to the results from the previous Subsection. We suppose that it is true for $p \leq n-1$ and prove it for $p = n$. We can proceed in strict analogy with the proof from Subsection 3.2. Everything stays unchanged with minor modification. The anomaly P_n is constructed in the same way and is constrained by the following conditions.

- It has the polynomial structure

$$P_n(x) = \sum_i p_i(\partial)\delta^{n-1}(x) W^Q_i(x); \tag{4.3.5}$$

here p_i are polynomials in the derivatives with the maximal degree restricted by

$$deg(p_i) + deg(W^Q_i) \leq 5 \tag{4.3.6}$$

and W_i are Wick monomials in all the free fields of the theory.

- $SL(2,\mathbb{C})$-covariance: it is unchanged

$$U_{a,A}P_n(x_1,\cdots,x_n)U_{a,A}^{-1} = T_n(\delta(A)\cdot x_1 + a,\cdots,\delta(A)\cdot x_n + a), \quad \forall (a,A) \in inSL(2,\mathbb{C}). \quad (4.3.7)$$

- PCT-covariance:

$$U_{PCT}P_n(x_1,\ldots,x_n)U_{PCT}^{-1} = (-1)^n P_n(-x_1,\ldots,-x_n). \quad (4.3.8)$$

- Unitarity:

$$P_n^\dagger \equiv (-1)^n P_n. \quad (4.3.9)$$

- Ghost numbers restrictions:

$$gh(P_n) = 1. \quad (4.3.10)$$

(ii) The list of possible anomalies (3.2.53) remains the same with the following amendment: the expressions $p_{\cdots}^{\cdots}$ are no longer numerical distribution but they can depend on the other fields $u_a,\ \tilde{u}_a,\ \Phi_a$ and $\psi_A,\ \overline{\psi_A}$. The degree limitations (3.2.54) are still valid if we mean by the degree of p the sum between the degree of the polynomial-type numerical distribution and the degree of the Wick monomial.

We now outline the analysis of these cases.

1) In this case the generic form of the distribution: p_{ijl}^{ρ} is not changed. This means that we have:

$$\mathcal{P}_1 = \delta^{n-1}(x) \quad \left[:\bar{\psi}_A(x_n)(K_a)_{AB}\gamma^\rho\psi_B(x_n): + :\bar{\psi}_A(x_n)(K'_a)_{AB}\gamma^\rho\gamma_5\psi_B(x_n):\right] \partial_\rho u_a(x_n). \quad (4.3.11)$$

But one proves rather elementary that from unitarity we have

$$K_a^* = (-1)^n K_a, \quad (K'_a)^* = (-1)^n K'_a \quad (4.3.12)$$

and from PCT-covariance

$$K_a^* = (-1)^{n-1} K_a, \quad (K'_a)^* = (-1)^{n-1} K'_a. \quad (4.3.13)$$

It follows that we must have $K_a = K'_a = 0$ i.e. $\mathcal{P}_1 = 0$.

2), 3) 4) Like in Subsection 3.2 we can reduce ourselves at the case:

$$\mathcal{P}_0(x) = \delta^{n-1}(x) \quad \left[:\bar{\psi}_A(x_n)(K_a)_{AB}\psi_B(x_n)::\bar{\psi}_A(x_n)(K'_a)_{AB}\gamma_5\psi_B(x_n):\right] u_a(x_n). \quad (4.3.14)$$

Now we get

$$K_a^* = (-1)^n K_a, \quad (K'_a)^* = (-1)^{n-1} K'_a \quad (4.3.15)$$

and from PCT-covariance

$$K_a^* = (-1)^{n-1} K_a, \quad (K'_a)^* = (-1)^n K'_a \quad (4.3.16)$$

so we can put to zero the corresponding anomalies.

5) In this case we get

$$\mathcal{P}_5(x) = \delta^{n-1}(x) \quad \left[:\bar{\psi}_A(x_n)(K_{ab})_{AB}\gamma^\rho\psi_B(x_n): + :\bar{\psi}_A(x_n)(K'_{ab})_{AB}\gamma^\rho\gamma_5\psi_B(x_n):\right] A_{a\rho}(x_n)u_b(x_n) \quad (4.3.17)$$

and we proceed as in case 1).

6) After integration by parts we get an expression of the type (3.2.56) with

$$W = \sum_{\alpha} W_{\alpha} \tag{4.3.18}$$

and

$$W_1 = c^1_{abcde} : \Phi_a\Phi_b\Phi_c\Phi_d : u_e \quad W_2 = c_{abcd} : \Phi_a\Phi_b\Phi_c : u_d \quad W_3 = c^1_{abc} : \Phi_a\Phi_b : u_c$$
$$W_4 = c^2_{abc} : \Phi_a\partial_\mu\Phi_b : \partial^\mu u_c \quad W_5 = c^1_{ab}\Phi_a u_b, \quad W_6 = c^2_{ab}\partial_\mu\Phi_a\partial^\mu u_b$$
$$W_7 = c_a u_a \quad W_8 = c^2_{abcde} : u_a u_b \tilde{u}_c :: \Phi_d\Phi_e : \quad W_9 = c^3_{abc} : u_a(\partial^\mu u_b)\tilde{u}_c : \tag{4.3.19}$$

One can show that from unitarity we have $c_{\dots} = (-1)^n c_{\dots}$ and from PCT-covariance $c_{\dots} = (-1)^{n-1} c_{\dots}$ so they must be zero.

7) In this case we have after integration by parts the formula (3.2.56) + (4.3.18) where

$$W_1 = c^1_{ab}(\partial_\mu A^\mu_a)u_b \quad W_2 = c^2_{ab}A^\mu_a\partial_\mu u_b \quad W_3 = c^3_{ab}(\partial^\nu A^\mu_a)(\partial_\mu\partial_\nu u_b)$$
$$W_4 = c^4_{ab}(\partial_\mu\partial_\nu A^\mu_a)(\partial^\nu u_b) \quad W_5 = c^1_{abc}(\partial_\mu A^\mu_a)u_b\Phi_c \quad W_6 = c^2_{abc}A^\mu_a(\partial_\mu u_b)\Phi_c$$
$$W_7 = c^3_{abc}A^\mu_a u_b(\partial_\mu\Phi_c) \quad W_8 = c^1_{abcd}(\partial_\mu A^\mu_a)u_b)\Phi_c\Phi_d$$
$$W_9 = c^2_{abcd}A^\mu_a(\partial_\mu u_b)\Phi_c\Phi_d \quad W_{10} = c^3_{abcd}A^\mu_a u_b(\partial_\mu\Phi_c)\Phi_d \quad W_{11} = c^4_{abcd}(\partial_\mu A^\mu_a)u_b\Phi_c\Phi_d. \tag{4.3.20}$$

This case is analyse exactly as the previous one.

8) We integrate by parts and end up with (3.2.56) + (4.3.18) where:

$$W_1 = c^1_{ab} : A^\mu_a\partial_\mu u_b : \quad W_2 = c^2 : (\partial^\rho A^\mu_a)(\partial_\rho\partial_\mu u_b) : \quad W_3 = c_{abc}A^\mu_a(\partial_{mu}u_b)\Phi_c$$
$$W_4 = c^1_{abcd}A^\mu_a(\partial_{mu}u_b)\Phi_c\Phi_d \quad W_5 = c^2_{abc}A^\mu_a(\partial_{mu}u_b)u_c\tilde{u}_d \tag{4.3.21}$$

Again, we can make $\mathcal{P}_8 = 0$ like in the previous case.

9) 10) We obtain (3.2.56) + (4.3.18) where:

$$W_1 = c^1_{abc}(\partial_\mu u_a) : (\partial^\nu A^\mu_b)A_{c\nu} : \quad W_2 = c^2_{abc}(\partial_\mu u_a) : (\partial^\mu A^\nu_b)A_{c\nu} : \quad W_3 = c^3_{abc}u_a : (\partial^\mu A^\nu_b)(\partial_\mu A_{b\nu}) :$$
$$W_4 = c^4_{abc}u_a : (\partial_\mu\partial_\mu A^\nu_b)(\partial^\nu A_{c\mu}) : \quad W_5 = c^5_{abc}u_a : A^\mu_b)A_{c\mu} : \quad W_6 = c_{abcd}u_a : A^\mu_b A_{c\mu} : \Phi_d$$
$$W_7 = c^1_{abcde}u_a : A^\mu_b A_{c\mu} : \Phi_d\Phi_e \quad W_8 = c^2_{abcde} : u_a u_b\tilde{u}_c :: A^\mu_d A_{e\mu}$$
$$W_9 = A_{abc}\varepsilon_{\mu\nu\rho\sigma}(\partial^\mu u_a) : (\partial^\nu A^\rho_b)A^\sigma_c : \tag{4.3.22}$$

The expressions $W_\alpha, \quad \alpha = 1,\dots 8$ are zero with the same argument as at 7) but the last term stays as it is for the moment.

11) 12) In these case we also have (3.2.56) + (4.3.18) where:

$$W_1 = c^1_{abcd}(\partial^\mu u_a) : A_{b\mu}A^\nu_c A_{d\nu} : \quad W_2 = c^2_{abcd}u_a : (\partial^\mu A^\mu_b)A^\nu_c A_{d\nu} : \quad W_3 = c^3_{abcd}u_a : (\partial^\nu A^\mu_b)A_{c\mu}A_{d\nu} : \tag{4.3.23}$$

which can be eliminated as at 7).

(iii) From the preceding computations it follows that the possible anomaly has the form

$$P_n(x) = \delta^{n-1}(x)A_{abc}\varepsilon_{\mu\nu\rho\sigma}(\partial^\mu u_a) : (\partial^\nu A^\rho_b)A^\sigma_c : \tag{4.3.24}$$

By hypothesis this anomaly does not appear in the order $p = 1,\dots,n-1 \quad (n \geq 4)$; we show that it does not appear for $p = n$. For this one must use lemma 3.4 and see that an expression of the type (4.3.24) can appear only for $n = 3$; for $n \geq 4$ we get factorizable distributions and we know that this sort of terms do nor appear in the expression of the anomaly.

It follows that we have succeed to prove the gauge invariance up to order $p = n$. If the resulting chronological product is not PCT-covariant, this can be mended as in [38] ch. 4.4. ■

Remark 4.4 *The last argument show that the anomalies do not get renormalized.*

Remark 4.5 *The non-uniqueness of the construction can be investigated as in the case of quantum electrodynamics with the same result.*

Remark 4.6 *The explicit condition of cancelation of the anomaly is investigated in [27] using also the Epstein-Glaser framework.*

Remark 4.7 *It is natural to expect that this approach to renormalization theory gives the same result as the usual procedure of Becchi, Rouet, Stora and Tyutin. A proof of this fact based on the quantum Noether method [33] appears in [3]. However, we can give here a much simpler argument. It is clear that both approaches verify the same set of axioms so one can use the characterization of the non-uniqueness given above.*

5 Conclusions

We have succeed to give complete proof of the renormalizability of the standard model. The proof is remarkable simple by comparison with the standard literature based on the usual BRST transformation (see [44] and literature quoted there). Its main advantages, beside the conceptual clearness, are:

1. the rôle of Feynman graphs is minimal (only in writing Wick theorem);
2. we emphasise the major rôle played by PCT-covariance;
3. we do not need the so called C-g identities [12], [13];
4. we do not need group theoretical arguments of the type appearing in [13];
5. we circumvent the problem of the so-called 1-particle reducible graphs (see the end of Subsection 3.2);
6. it appears that if one has well understood the renormalization of quantum electrodynamics, then the case of the standard model follows almost as a corollary.

It is to be expected that this method simplifies the analysis of other gauge models of physical interest from the literature. We propose to do this in future papers.

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